Capacitated SIR Model with an Application to COVID-19

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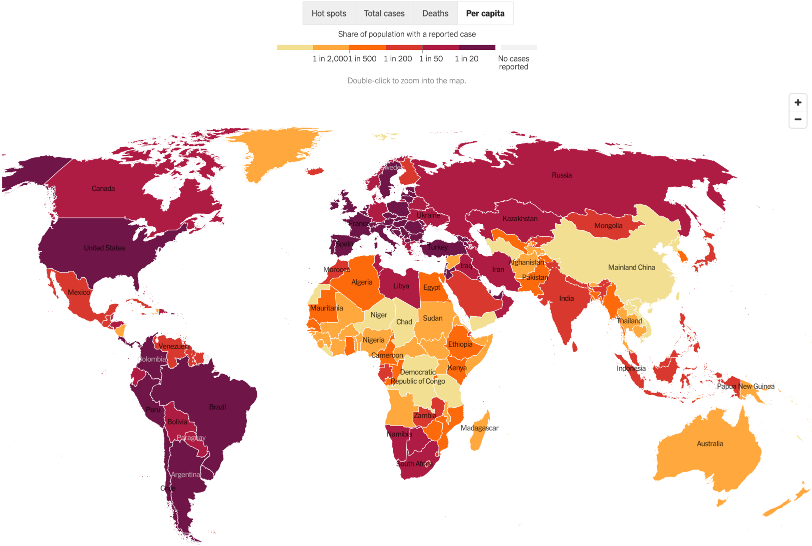
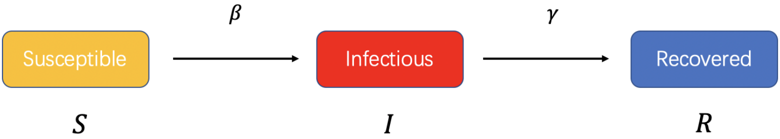
COVID-19


Hot spots
Total cases
Deaths
Per capita
Share of population with a reported case
1 in 2,000
1 in 500
1 in 200
1 in 50
1 in 20
No cases reported
Double-click to zoom into the map.
Canada
United States
Mexico
Venezuela
Colombia
Brazil
Peru
Bolivia
Paraguay
Argentina
Chile
Russia
Sweden
Ukraine
France
Spain
Turkey
Kazakhstan
Mongolia
Mainland China
Morocco
Algeria
Libya
Egypt
Iraq
Iran
Afghanistan
Pakistan
India
Mauritania
Niger
Chad
Sudan
Nigeria
Ethiopia
Cameroon
Kenya
Democratic Republic of Congo
Zambia
Namibia
South Africa
Madagascar
Thailand
Indonesia
Papua New Guinea
Australia

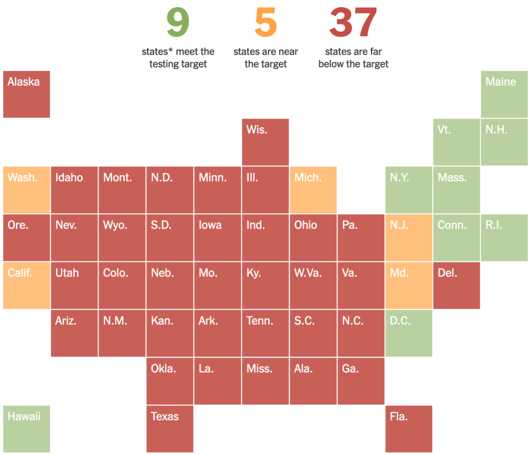
SIR Model



$$\frac{dS}{dt} = -\frac{\beta IS}{N}$$
$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Testing at Early Stage

On November 1, 2020:



Testing Nowadays

Now testing still matters!

- Across US, attention shifted from testing to vaccination.

  Obviously, vaccines are quite important. But as long as the majority of us are not protected, then testing remains essential.

  —Jennifer Nuzzo at JHU

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Benefits:

- Diagnosing sick people
- Slowing the spread
- Assessing our progress

Literature Review

Transition modeling:

- Acemoglu, Chernozhukov, Werning, Whinston (2020)
- Birge, Candogan, Feng (2020)
- Henderson, Shmoys, Frazier (2020)

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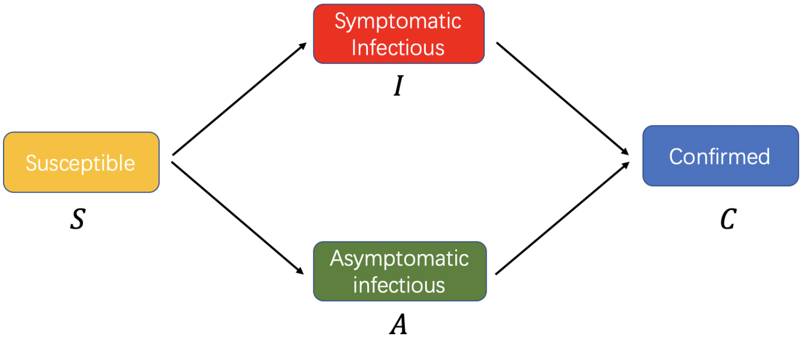
Testing capacity:

- Acemoglu, Makhdoumi, Malekian, Ozdaglar (2020)
- Housni, Sumida, Rusmevichientong, Topaloglu, Ziya (2020)
- Berger, Herkenho, Mongey (2020)
- Larson, Berman, Nourinejad (2020)

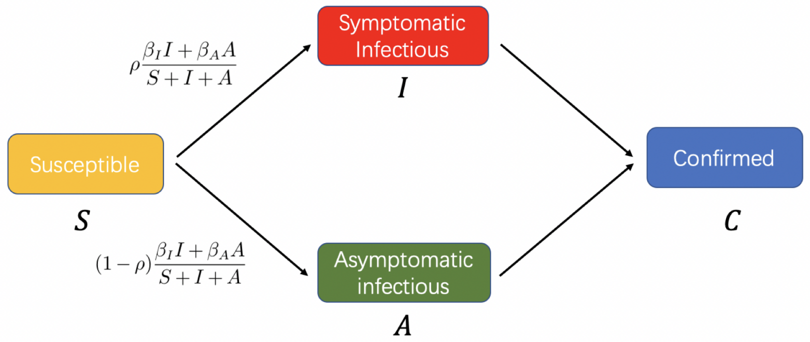
Setup

A compartmental model with four states:



Setup

A compartmental model containing four states:



Setup

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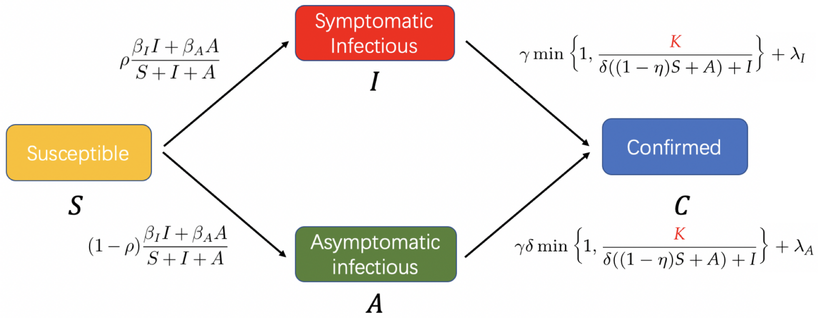


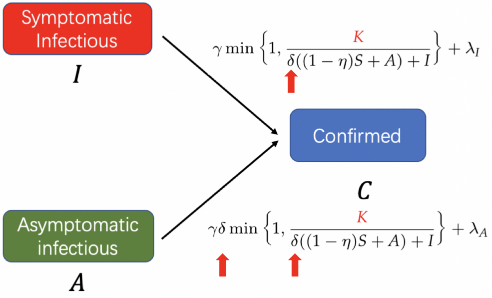
System Dynamics

$$
\begin{aligned}
S_{t+1} &= (1 - \frac{\beta_I I_t + \beta_A A_t}{S_t + I_t + A_t})S_t \\
I_{t+1} &= \frac{\beta_I I_t + \beta_A A_t}{S_t + I_t + A_t}\rho S_t + \left(1 - \gamma \min\left\{1, \frac{K_t}{\delta((1-\eta)S_t + A_t) + I_t}\right\} - \lambda_I\right) I_t \\
A_{t+1} &= \frac{\beta_I I_t + \beta_A A_t}{S_t + I_t + A_t}(1-\rho)S_t + \left(1 - \gamma\delta \min\left\{1, \frac{K_t}{\delta((1-\eta)S_t + A_t) + I_t}\right\} - \lambda_A\right) A_t \\
C_{t+1} &= \left(\gamma \min\left\{1, \frac{K_t}{\delta((1-\eta)S_t + A_t) + I_t}\right\} + \lambda_I\right) I_t \\
&\quad + \left(\gamma\delta \min\left\{1, \frac{K_t}{\delta((1-\eta)S_t + A_t) + I_t}\right\} + \lambda_A\right) A_t + C_t.
\end{aligned}
$$

Testing Resource Allocation

- Allocation of testing resources among different populations.
- Allocation of testing resources across time.
  - Two periods.
  - Multi-periods.

Allocation of Testing Resources Among Populations



Proposition

There exists a threshold $h \in [\frac{1-s_0-\rho(1-s_0)}{1-\eta s_0-\rho(1-s_0)}, 1]$, *such that*

1. *when* $\frac{\beta_I}{\beta_A} \leq h$, *it is optimal to allocate more testing resources to people without symptoms.*
2. *when* $\frac{\beta_I}{\beta_A} > h$, *it is optimal to allocate more testing resources to people with symptoms.*

Allocation Across Time–Two Periods

Proposition

There exists a threshold $\frac{S_0+I_0+A_0}{I_0+A_0} \geq 1$*, such that*

1. *if* $\min\{\beta_I, \beta_A\} \geq \frac{S_0+I_0+A_0}{I_0+A_0}$*, it is optimal to allocate all the testing resources to period 2.*
2. *if* $\max\{\beta_I, \beta_A\} < \frac{S_0+I_0+A_0}{I_0+A_0}$*, it is optimal to allocate all the testing resources to period 1.*

Allocation Across Time: Multi-Periods

Dynamic programming:

- Time horizon: τ days.
- Optimal value function: $v^*(S_n, I_n, A_n, k_n)$.
- Bellman equation:

$$v^*(S_n, I_n, A_n, k_n) = \min_{\kappa}\{v^*(S_{n+1}, I_{n+1}, A_{n+1}, k_n - \kappa) + (\frac{\gamma m}{\delta((1-\eta)S_n + A_n) + I_n} + \lambda_I)I_n$$

$$+ (\frac{\gamma \delta m}{\delta((1-\eta)S_n + A_n) + I_n} + \lambda_A)A_n + 1 - S_n - I_n - A_n\}$$

Allocation Among Different Time: Multi-

where

$$
\begin{aligned}
S_{n+1} &= (1 - \frac{\beta_I I_n + \beta_A A_n}{S_n + I_n + A_n}) S_n, \\
I_{n+1} &= \rho \frac{\beta_I I_n + \beta_A A_n}{S_n + I_n + A_n} S_n + (1 - \frac{\gamma m}{\delta((1-\eta)S_n + A_n) + I_n} - \lambda_I) I_n, \\
A_{n+1} &= (1-\rho) \frac{\beta_I I_n + \beta_A A_n}{S_n + I_n + A_n} S_n + (1 - \frac{\gamma \delta m}{\delta((1-\eta)S_n + A_n) + I_n} - \lambda_A) A_n, \\
k_0 &= C, \\
v^*(S_0, I_0, A_0, k_0) &= 0, \\
v^*(S_\tau, I_\tau, A_\tau, k_\tau) &= 0, \\
n &\in \{1, \tau - 1\}.
\end{aligned}
$$

Ranking of Testing Methods

$$\text{testing index} = \frac{\text{testing capacity} \times \text{testing accuracy}}{\text{testing turnaround time}}.$$

Table: Summary statistics for two testing methods using the data of Ontario, Canada

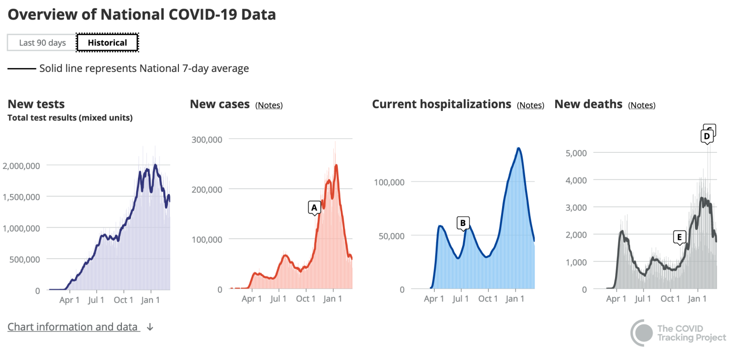
Testing types	Testing capacity	Testing accuracy	Expected testing turnaround time	Testing index
Rapid tests	9,681,629	False negative: 15%	15 mins	$7.90e10$
Molecular tests	20,128,734	False negative: 5%	1.5 days	$1.27e7$

Summary of structural results

Parameter	Interpretation	Structural result on S_2	Policy implication
K	Testing capacity	Increasing and convex	Testing capacity expansion ($K \uparrow$)
δ	Degree of testing people without symptoms and panic run	When $\frac{\beta_I}{\beta_A} \geq h$, decreasing and convex	Contact tracing, and self-quarantine, stay at home unless severely ill ($\delta \downarrow$)
γ	Testing accuracy divided by testing turnaround time	Increasing and convex	Testing turnaround time shortening, testing accuracy increasing ($\gamma \uparrow$)
η	Tracing accuracy	Increasing and convex	Contact tracing, case investigation, quarantine ($\eta \uparrow$)
β_I	Infection rate of infected and symptomatic population	Decreasing	Social distancing, mask-wearing, quarantine ($\beta_I \downarrow$)
β_A	Infection rate of infected and asymptomatic population	Decreasing	Social distancing, mask-wearing, quarantine ($\beta_A \downarrow$)

Data

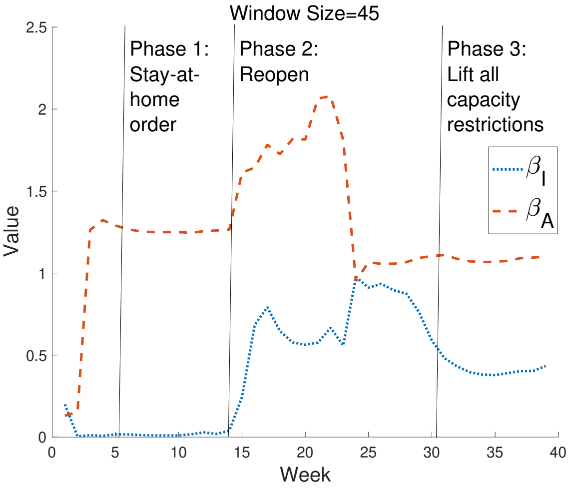
- Use the COVID-19 data from the COVID Tracking Project.
- Use the daily number of completed viral tests and cumulative confirmed cases from this dataset.



Estimation

- Use the sliding window method.
- Extract K_t (the number of tests on day t) and $C_t - \lambda_I I_t - \lambda_A A_t$ (the number of confirmed cases on day t).
- Can identify the changes of model parameters.

Estimated Parameters for Florida in the U.S.



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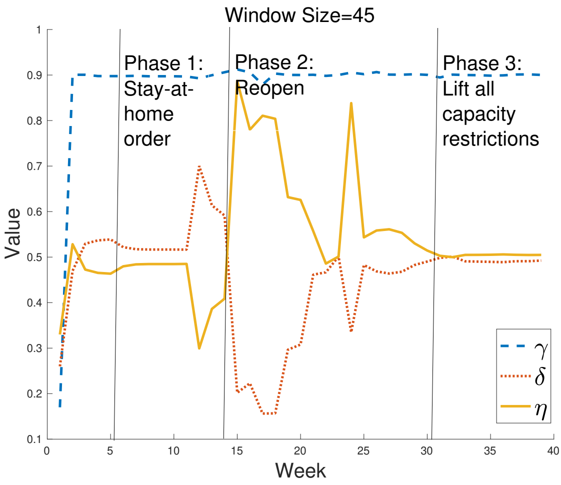

Summary

- Propose a capacitated SIR model.
- Provide policy implications like how to allocate testing resources and how to choose testing methods.
- Using sliding window method in empirical studies to test the changes of parameters.

Thank you!

Available at SSRN: `https://ssrn.com/abstract=3692751`