Bridging the Gap between Strategic Allocation and Investment Risk

Redouane Elkamhi, Jacky S.H. Lee, and Sheikh Sadik

Redouane Elkamhi
is an associate professor of finance at the Rotman School of Management at the University of Toronto in Toronto, Ontario, Canada.
redouane.elkamhi@rotman.utoronto.ca

Jacky S.H. Lee
is vice president of total portfolio at the Healthcare of Ontario Pension Plan Trust Fund in Toronto, Ontario, Canada.
jlee5@hoopp.com

Sheikh Sadik
is a researcher at the Rotman School of Management at the University of Toronto in Toronto, Ontario, Canada.
sheikh.sadik@mail.utoronto.ca

KEY FINDINGS

- The authors propose a methodology to address the common inconsistency that exists between models used for long-term strategic asset allocation and investment risk management.
- The authors show that it is possible to reflect long-term asset characteristics in simulated scenarios generated by a risk system calibrated with short-term history, creating a better alignment between risk and strategic asset allocation models.
- The methodology allows institutional investors to better use existing simulations from their risk models for portfolio allocation, sensitivity analysis, stress testing, and other portfolio applications.

ABSTRACT

For many institutional investors, there is a potential inconsistency between models used for long-term strategic asset allocation and investment risk management. Investment risk models, often calibrated with a shorter history spanning 5 to 15 years, could provide misleading results when used for strategic portfolio construction decisions, which usually consider longer-term asset characteristics spanning multiple business cycles. In this article, the authors propose a methodology to address this challenge. They show that it is possible to reflect long-term asset characteristics in simulated scenarios generated by a risk system calibrated with short-term history, creating a better alignment between risk and strategic asset allocation models. Their methodology allows institutional investors to better use existing simulations from their risk models for portfolio allocation, sensitivity analysis, stress testing, and other portfolio applications.

TOPICS

Portfolio construction, quantitative methods, simulations, risk management, performance measurement*

*All articles are now categorized by topics and subtopics. **View at PM-Research.com**.

Institutional investors with a long-term focus usually achieve their return objectives through a combination of strategic asset allocation and a value-added program. These investment decisions naturally involve making trade-offs between risks and returns. To properly manage the overall investment risk of a fund, sound risk management practices are established for both the strategic asset allocation (total risk) and the value-added program (active risk). In practice, usually a single system is used to estimate and govern the investment risk of both activities. Modern risk management systems face numerous constraints and requirements, with a key consideration being data availability. Because each position is prudently modeled in a risk system involving tens of thousands of financial securities, the modeling period is often restricted to short-term history (typically 5 to 15 years) because of the demanding data requirements. The situation is the opposite for strategic portfolio construction modeling. Long-term history (50 years or more, spanning multiple business cycles) is often needed to calibrate strategic portfolio construction models involving fewer securities. The aforementioned situation creates a potential inconsistency between strategic portfolio construction and risk management models.

To illustrate the economic significance of this inconsistency, let's consider the empirical relationship between equity and fixed income. This pairwise correlation for the 2015–2020 period has been negative, resulting in considerable diversification benefits between these two asset classes. However, going back further in history (e.g., since 1950), the pairwise correlation has been close to zero. A straightforward implication of this correlation difference is that a risk-reducing asset allocation shift decided using short-term history could instead be risk-adding when analyzed using long-term history. The difference in modeling period between the risk and strategic portfolio construction models could adversely affect portfolio management decisions. The aim of this article is to propose a simple methodology on how a risk system, although calibrated with short-term history, can be leveraged appropriately to be more aligned with long-term strategic asset allocation.

Our approach is inspired by the change of measure and importance sampling. We show that it is possible to simply reweight scenarios simulated by a risk system to reflect the different asset characteristics. The following provides a quick overview of the methodology.

Given a set of simulated risk scenarios, several relevant factors are selected, and their simulated joint distribution is estimated. Next, the preferred joint distribution of these factors is determined by using a different modeling period or forward-looking views. Finally, the likelihood weights are computed using the preferred and simulated factor distributions and are applied to the scenarios, thus altering the probability distributions of all simulated asset returns. The altered joint distribution of simulated returns provides an alternative view on portfolio and factor statistics such as returns, volatility, expected shortfall, and so forth.

We demonstrate the practicality of our methodology through three real-world examples related to investment and risk management applications, including portfolio what-if analysis on modeling assumptions and portfolio stress testing. The first example directly addresses the inconsistency raised earlier. We create a short-term risk model on 18 assets and define three factor portfolios. We apply our methodology to reflect a different joint distribution of the factors based on long-term history and demonstrate that we are able to reflect long-term asset characteristics on the scenarios generated by the short-term risk model. In the second example, we analyze the portfolio volatility impact under different equity–fixed income correlation assumptions. This is achieved by applying our methodology to reflect different equity–fixed income correlations and investigating the resulting likelihood-weighted portfolio volatilities. The results of this analysis can help investors determine the appropriate level of portfolio risk under different correlation assumptions. The third example illustrates the application of portfolio stress testing using simulated scenarios. In common practice, portfolio stress testing involves repricing a portfolio under user-defined market shocks. Using our methodology, we define these market shocks as target probabilities and use them to compute the likelihood weights. These weights are then applied to the risk scenarios to calculate stressed portfolio returns.

Overall, we show using three examples that our methodology allows institutional investors to better use existing simulations from their risk models for portfolio allocation, sensitivity analysis, and stress testing with alternative or forward-looking views, thus bridging the gap between investment risk measurement and strategic allocation.

REVIEW OF PREVIOUS LITERATURE

This article contributes to the large literature on long-term asset allocation and investment risk management. Many important contributions have been made in the areas of optimal portfolio allocation, risk, and factors (see Markowitz 1952; Black and Litterman 1990; Qian 2011; Asness, Frazzini, and Pedersen 2012). However, practical issues arise during the implementation of these theories. First, strategic asset allocation using these approaches requires the models to be calibrated based on long-term history owing to investors' strategic objectives and the notorious difficulty in estimating expected returns and correlation through the cycle. Second, and more importantly, the assumptions made in the strategic asset allocation model are rarely aligned in practice with the assumptions made in the underlying risk system.

To elaborate further, it is widely known that modern risk systems face regulatory, operational, and governance requirements as well as business demands such as system functionalities, product coverage, reporting, and processes (Gillespie and Curwood 2012). These demanding requirements make risk models different from strategic asset allocation models. For insurance companies and pension plans, asset–liability models are the primary tool for strategic asset allocation. However, for value-added programs, these institutions use modern risk systems to perform risk budgeting and monitoring (Franzen 2010). Our article contributes to this extant literature by proposing a methodology that could create a better alignment between modern risk management and strategic asset allocation when needed.

The methodology we present is inspired by the change of measure and importance sampling. The change of measure is applied extensively to derivatives pricing; see Shreve (2004), Margrabe (1978), Geman, El Karoui, and Rochet (1995), Schroder (1999), and Benninga, Bjork, and Wiener (2001) for more details. As for importance sampling, it is typically used as a variance reduction technique in derivatives pricing, portfolio risk modeling, and signal processing; see Glasserman (2003) for more details. Our approach builds on previous research and instead focuses on addressing an important challenge related to strategic asset allocation and risk management, which is novel in the extant literature.

METHODOLOGY

Our methodology is centered on determining the multivariate likelihood ratio between the desired and simulated joint probability distribution of asset returns. We start this section by expressing the multivariate asset return distribution generated by a risk model as a copula and marginal distributions of risk factors. We simplify the expression using the law of conditional probability to separate out the few selected factors of interest. We then derive the likelihood ratio between the desired and simulated joint distribution on these factors. The section concludes by explaining the regularity conditions for likelihood ratios and showing its linkage to the change of measures.

Main Derivation

Let the underlying factors of a risk model follow a general class of multivariate distribution, denoted by $p(x_1, \ldots, x_n)$, with $Z \equiv \{x_1, x_2, \ldots, x_n\}$ being a set of n risk factors. According to Sklar's (1959) theorem, there must exist a unique copula that links the risk factor marginal distributions to create a joint probability distribution:

$$p(x_1, \ldots, x_n) = c(\Phi_1(x_1), \ldots, \Phi_n(x_n))\prod_{i=1}^{n}\phi_i(x_i) \tag{1}$$

where $\Phi(*)$ and $\phi(*)$ are the cumulative and probability density functions of the individual risk factors, respectively; $C(*)$ and $c(*)$ are the cumulative and probability density functions of the copula, respectively. Let's partition the n risk factors into two subsets, denoted by Z_A and Z_B, where $Z = Z_A \cup Z_B$, $Z_A \equiv \{x_1, x_2, x_3, \ldots, x_m\}$, and $Z_B \equiv \{x_{m+1}, x_{m+2}, x_{m+3}, \ldots, x_n\}$ with $m < n$. The multivariate distribution of all risk factors $p(Z_A, Z_B)$ can be expressed as $p(Z_A, Z_B) = p(Z_B|Z_A)p(Z_A)$ using the law of conditional probability. This provides the ability to modify the selected marginal joint distribution and not the entire joint distribution. We denote Z_A as the subset of risk factors we are interested in altering and Z_B the subset of risk factors whose joint distribution conditional on Z_A we want to preserve.

Let's define the joint distribution of the risk factor scenarios as $f(Z)$ and the desired target joint distribution as $g(Z)$. The target distribution incorporates the risk factor characteristics based on user inputs (e.g., a different correlation matrix or asset volatilities). We can write the likelihood ratio$l$between the two distributions as follows:

$$l(Z) = \frac{g(Z)}{f(Z)} = \frac{p^g(Z_A, Z_B)}{p^f(Z_A, Z_B)} = \frac{p^g(Z_B|Z_A)p^g(Z_A)}{p^f(Z_B|Z_A)p^f(Z_A)} \tag{2}$$

Furthermore, we set the conditional probabilities to be the same under both measures; that is, $p^g(Z_B|Z_A) = p^f(Z_B|Z_A)$. Under this assumption, the likelihood ratio can be approximated as

$$\begin{aligned} l(Z_A) = \frac{p^g(Z_A)}{p^f(Z_A)} &= \frac{c^g(\Phi_1^g(x_1), \ldots, \Phi_m^g(x_m))\prod_{i=1}^{m}\phi_i^g(x_i)}{c^f(\Phi_1^f(x_1), \ldots, \Phi_m^f(x_m))\prod_{i=1}^{m}\phi_i^f(x_i)} \\ &\cong \frac{c^p(\Phi_1^g(x_1), \ldots, \Phi_m^g(x_m) \mid \Theta_1)\prod_{i=1}^{m}\phi_i^g(x_i)}{c^p(\Phi_1^f(x_1), \ldots, \Phi_m^f(x_m) \mid \Theta_0)\prod_{i=1}^{m}\phi_i^f(x_i)} \end{aligned} \tag{3}$$

where $c^p(*)$ is a parametric copula used to represent $c^g(*)$ and approximate $c^f(*)$; Θ_0 is the maximum likelihood estimated parameters of the copula under probability measure f; and Θ_1 is the target copula parameters defined under the new measure g. Equation 3 assumes the parametric copula is a good representation of the empirical copula such that $c^f(\Phi_1^f(x_1), \ldots, \Phi_m^f(x_m))/c^p(\Phi_1^f(x_1), \ldots, \Phi_m^f(x_m)|\Theta_0) \cong 1$ for all values of x. A full derivation is provided in online Appendix A.

Regularity Conditions, Likelihood Weights, and Effective Sample Size

Under the probability measure $f(Z)$, we can define a likelihood function $l(Z) = dP^g(Z)/dP^f(Z)$, which is also the Radon–Nikodym derivative of g with respect to f. For g to be a probability measure, the likelihood ratio l must be ≥0. Furthermore, the probability measure g must be absolute continuous with respect to the measure f. From a numerical standpoint, we need the absolute continuity to hold. This means the scenario generated under distribution f must be extensive enough to be sampled from the new distribution g for our approach to work desirably. In a discrete setting, we can express the likelihood weight $w(Z_A)$ for scenario j as follows:

$$w(Z_A)_j = \frac{l(Z_A)_j}{\sum_{j=1}^{S} l(Z_A)_j} \tag{4}$$

where S is the total number of scenarios, $0 \le j \le S$, $w(Z_A)_j \ge 0$ with $\Sigma_j^S w(Z_A)_j = 1$. Notice that the likelihood ratio is only a function of the scenario index j, which means the same weight value is applied to all asset and factor returns for each scenario. These weights are akin to the likelihood weights originally proposed in Tim Hesterberg's (1988) thesis. Detailed derivation of this section is provided in online Appendix B.

Reweighting a set of scenarios is analogous to using only a subset but repeated scenarios; thus, we can expect information loss as the number of "effective" samples decreases. To assess the amount of information that is retained, we calculate the effective sample size (ESS) derived by Kish (1965). ESS is defined as

$$ESS = \frac{\left(\sum_{j=1}^{S} w(Z_A)_j\right)^2}{\sum_{j=1}^{S} w(Z_A)_j^2} = \frac{1}{\sum_{j=1}^{S} w(Z_A)_j^2} \tag{5}$$

EMPIRICAL ANALYSIS

In this section, we outline in detail the first example briefly mentioned in the introduction. The motivation of this example is to demonstrate the relevance of our methodology in a real-world application. We create a straightforward risk model calibrated with recent history to generate simulated scenarios of asset returns. We apply our methodology to alter the joint distribution of these asset return scenarios to reflect the asset characteristics over a longer history. To assess the effectiveness of the methodology, we recalibrate the risk model with a longer history to serve as a benchmark for comparisons. We then compare and evaluate portfolio metrics such as risk contribution and asset-to-portfolio correlations between the weighted scenarios and the benchmark scenarios.

Portfolio Setup and Implementation

We source data from Global Financial Data and Bloomberg. The list of indices and the details of data preparation are provided in online Appendix C. The dataset includes seven equity (EQ) indices (Australia [AU], Canada [CA], France [FR], Germany [DE], the United Kingdom [UK], Japan [JP], and the United States [US]), seven fixed income (FI) indices (Australia [AU], Canada [CA], France [FR], Germany [DE], the United Kingdom [UK], Japan [JP], and the United States [US]), US Corporate Credit (CORP-US), and three commodity (COM) indices (energy [EN], agriculture [ARG], and precious metals [PM]).

For our example risk model, we apply the stationary bootstrap algorithm by Politis and Romano (1994) on the historical returns. We simulate two sets of 10,000 asset return scenarios using data from January 2004 to December 2018 (representing a short-term risk model) and from February 1971 to December 2018 (representing the long-term benchmark model). The benchmark model, although feasible in this demonstration, is impractical for an actual investment risk system because of a lack of data for most securities dating back to 1971. The simulated scenarios are demeaned before comparison analysis is performed.

EXHIBIT 1

Comparisons of Asset and Portfolio Risk Statistics between the Short- and Long-Term Risk Scenarios and Likelihood Weighted Scenarios

		[Sim A] Original Short Term: 2004–2018		[Sim B] Target Long Term: 1971–2018		[Sim C] Weighted Short Term: 2004–2018		[Sim D] Weighted Short Term: 2004–2018	
		Equal Weights		Equal Weights		Weighted by Factor Set [m_1]		Weighted by Factor Set [m_2]	
1% ES		–24.0		–22.2		–22.2		–21.7	
ESS %						50.5		48.7	
Asset ID	Portfolio Weight (%)	1% ES Contr.	ρ_P	1% ES Contr.	ρ_P	1% ES Contr.	ρ_P	1% ES Contr.	ρ_P
EQ-AU	3.2	5.7	81.5	6.2	69.0	5.2	78.7	5.1	78.1
EQ-CA	3.2	6.3	86.8	6.3	81.0	5.5	83.4	5.4	82.1
EQ-FR	3.2	6.0	78.0	6.2	66.2	5.3	73.9	5.3	73.9
EQ-DE	3.2	6.7	78.6	5.6	60.1	6.0	75.3	6.0	75.6
EQ-UK	3.2	5.1	85.8	5.7	66.1	4.9	83.9	4.9	83.8
EQ-JP	4.0	8.2	59.2	6.3	51.7	7.0	52.4	7.0	51.8
EQ-US	20.0	38.3	87.7	37.3	84.3	34.4	84.9	34.4	85.3
FI-AU	4.0	–1.5	–24.5	0.0	19.5	–0.2	10.7	–0.2	12.1
FI-CA	4.0	–0.2	–4.1	0.8	34.4	0.8	29.0	0.8	30.0
FI-FR	4.0	–0.5	–0.4	0.3	26.4	0.6	33.5	0.6	34.2
FI-DE	4.0	–0.8	–13.5	0.1	22.6	0.2	22.7	0.3	24.0
FI-UK	8.0	–1.3	–5.7	1.0	25.5	1.3	30.2	1.4	32.3
FI-JP	4.0	–0.1	–1.5	0.4	22.0	0.3	22.1	0.3	22.6
FI-US	12.0	–2.1	–8.7	0.9	22.7	2.5	26.2	2.8	28.5
CORP-US	5.0	2.0	50.6	2.1	47.1	3.0	65.0	3.0	65.1
COM-EN	5.0	16.9	58.5	10.8	28.1	12.0	33.7	11.7	30.5
COM-ARG	5.0	7.6	49.8	5.2	29.7	6.6	35.8	6.6	34.5
COM-PM	5.0	3.6	32.4	4.6	28.3	4.7	30.0	4.7	27.5

NOTES: The exhibit reports key risk metrics and statistics for four different risk simulations. Sim A and Sim B are scenarios (equal weights) based on the short- (2004–2018) and long-term (1971–2018) risk models, whereas Sim C and Sim D are scenarios from the short-term risk model weighted based on our methodology incorporating the long-term characteristics of factor portfolios m_1 and m_2 defined in the online Appendix, Exhibit C2. 1% ES denotes the 1% portfolio expected shortfall, 1% ES Contr. is the portfolio expected shortfall contribution from each asset, ρ_p is the correlation between the portfolio and each asset, and ESS % is the effective sample size in percentage for Sim C and Sim D. The table shows that the metrics from Sim C and Sim D are close in most aspects to Sim B. This demonstrates the effectiveness of our methodology.

In this example, we introduce three systematic factor portfolios (equity, fixed income, and inflation) and a hypothetical portfolio. The equity, fixed income, and inflation factors are built from the seven country equity indices, the seven country fixed income indices, and the three commodity indices, respectively. We are interested in testing the impact of factor composition; therefore, we create two factor sets. The detailed asset weights of each factor set are listed in online Appendix Exhibit C2. In a quick summary, factor set 1 is constructed with equal weights within each asset class, whereas factor set 2 uses our test portfolio sub-asset class weights. The weights of the hypothetical portfolio are shown in Exhibit 1.

After the risk simulation, we calculate the factor scenarios based on the asset weights of these factor portfolios. We then calculate the marginal probability densities of the short-term ϕ^f and long-term ϕ^g factor scenarios using a normal kernel density estimator. To compute the short-term c^f and long-term c^g copula densities, we use the Student's t-copula as a proxy copula and estimate its parameters via maximum likelihood estimation. With these marginal and copula densities, we compute the likelihood weights. In practice, if we know the risk system uses the Gaussian copula, then it could be used for c^f and c^g; otherwise, a Student's t-copula would provide more flexibility in fitting the empirical copula of the simulation.

Discussion of Results

In this section, we present our results and the key findings that relate to the first example described previously. There are four test cases for comparison: simulation from the short- (Sim A) and long-term (Sim B) risk models and the weighted simulations from the short-term risk model with likelihood weights computed with factor set m_1 (Sim C) and m_2 (Sim D), respectively. Exhibits 1 and 2 report the portfolio weights, the 1% expected shortfall (ES), asset–portfolio correlation, and the portfolio expected shortfall. See online Appendix D for further details on why we choose to demonstrate the usefulness of our methodology by focusing on the asset–portfolio correlation.

There are several important observations in Exhibits 1 and 2. Note that Sim A and Sim B show different portfolio risk profiles. Examining them in Exhibit 1 shows that the role of fixed-income assets is different for this hypothetical portfolio. Based on the short-term model, fixed-income assets offer great diversification benefit, as suggested by the zero to slightly negative risk contributions. However, the long-term model suggests fixed-income assets offer comparatively less diversification benefit, as shown by the significantly higher asset–portfolio correlation. Observing Sim C and Sim D compared to Sim B (the benchmark) reveals encouraging results on the performance of our methodology. The asset–portfolio correlations of fixed-income assets are similar among Sim B, Sim C, and Sim D but not Sim A. The 1% expected shortfall contribution and the asset–portfolio correlations are reasonably close among Sims B, C, and D for most assets. Examining Exhibit 2 on the factor correlations corroborates these conclusions. This finding supports the notion that the weighted risk scenarios

EXHIBIT 2

Factor Correlations between the Short- and Long-Term Risk Scenarios and Likelihood Weighted Scenarios

		[Sim A] Original Short Term: 2004–2018				[Sim B] Target Long Term: 1971–2018				[Sim C] Weighted Short Term: 2004–2018				[Sim D] Weighted Short Term: 2004–2018			
		Equal Weights				Equal Weights				Weighted by Factor Set [m_1]				Weighted by Factor Set [m_2]			
Factor Set		F_1	F_2	F_3	P	F_1	F_2	F_3	P	F_1	F_2	F_3	P	F_1	F_2	F_3	P
m_1	F_1																
	F_2	–41.3				2.6				–3.0							
	F_3	43.0	–29.4			11.0	–22.7			9.0	–19.6						
	P	87.1	–10.4	67.3		84.7	33.6	38.0		83.8	27.9	43.5					
m_2	F_1																
	F_2	–41.2				3.4								–1.7			
	F_3	43.5	26.1			9.1	–22.3							7.4	–17.7		
	P	88.9	–9.6	67.2		87.7	32.3	38.0						85.9	30.1	40.3	

NOTES: The exhibit reports the pairwise factor (F_1, F_2, and F_3) correlations (in percent) and the factor correlations to the portfolio (P) for factor sets m_1 and m_2 on four different risk simulations. Sim A and Sim B are scenarios (equal weights) based on the short- (2004–2018) and long-term (1971–2018) risk models, whereas Sim C and Sim D are scenarios from the short-term risk model weighted based on our methodology incorporating the long-term characteristics of factor portfolios m_1 and m_2 defined in online Appendix Exhibit C2.

could be used for long-term asset analysis, including potentially complementing existing strategic asset allocation models. This could be particularly important when asset allocation decisions involve assets with different relationships at different points in history, such as equity and fixed income. Finally, if Sim B is indeed the desired simulation for long-term asset allocation and portfolio analysis, then a surrogate of Sim B could be obtained by applying our methodology to Sim A.

Other Considerations

So far, we have discussed how we can reweight the simulated scenarios using a set of systematic factors. When implementing our methodology, it is important to choose factors that explain a good portion of the portfolio risk. Factors that do not explain the portfolio risk sufficiently will not yield meaningful results. See online Appendix E for further details on the importance of factor choices in our methodology.

To demonstrate the statistical validity of our methodology, we conduct a set of controlled experiments. We show that our methodology works well in altering the marginal and codependence structure of a known distribution family for a range of parameters. We choose the Student's *t*-distribution and the Student's *t*-copula to perform the experiment because they can generate a range of tail distributions and dependencies, including the Gaussian case when the degree of freedom is large. These tests demonstrate that when the properties of the Radon–Nikodym theorem are satisfied, the joint distribution of the weighted scenarios is consistent with the target probability measure. To save space, the controlled experiments are reported in online Appendix F and corresponding online Appendix Exhibits F1 to F5.

OTHER APPLICATIONS

In this section, we discuss the portfolio sensitivity analysis and stress testing applications briefly mentioned in the introduction. These two examples are inspired by the desire of many investment managers to perform what-if portfolio analysis on different market conditions and model assumptions.

Portfolio Sensitivity Analysis

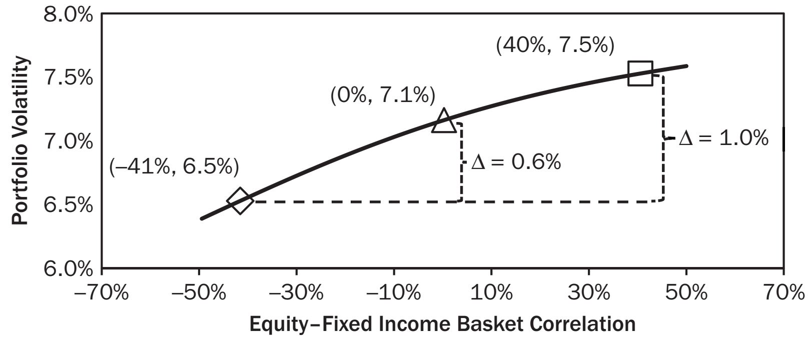
This example demonstrates an application to quantify the impact to portfolio volatility under different equity–fixed income factor correlation assumptions. Exhibit 3 plots the portfolio volatility from Sim A of Exhibit 1 with varying degrees of correlation between the equity and fixed income factors. The portfolio volatility is calculated from the weighted risk scenarios after our methodology is applied to alter the factor correlations. As shown in Exhibit 3, the portfolio volatility changes by 0.6% and 1.0% if the equity–fixed income factor correlation changes from −41% to 0% and 40%, respectively. The key finding is that the risk of a balanced portfolio estimated using recent history could be too low because of the significant negative equity–fixed income factor correlation assumed in the model. This type of application opens up a continuum of possibilities for more analysis regarding the sensitivity of different modeling assumptions to asset allocation and risk measurement.

Stress Testing

This practical example relates to portfolio stress testing. Using our methodology, we calculate the likelihood weights based on a few selected factors or assets that

EXHIBIT 3

Portfolio Volatility as a Function of Equity–Fixed Income Correlation



NOTES: Based on the setup from the Empirical Analysis section, this graph shows the expected portfolio volatility for varying $F_1 - F_2$ (equity–fixed income) factor correlation from –50% to +50%. The diamond denotes the model portfolio volatility from Sim A of Exhibit 1, with an $F_1 - F_2$ correlation of –41%. The triangle and the square denote the model portfolio volatility with the $F_1 - F_2$ correlation changed to 0% and +40%, respectively.

are put under stress. For each scenario j, we design the target distribution and write the likelihood ratio as

$$l(Z)_j = \frac{\prod_{i=1}^{m} N\left(\frac{x_{i,j} - \mu_i}{\alpha\sigma_i}\right)}{c^p(\Phi_1^f(x_{1,j}), \ldots, \Phi_m^f(x_{m,j})|\Theta_0)\prod_{i=1}^{m}\phi_i^f(x_{i,j})} \tag{6}$$

where $N(*)$ is the standard Gaussian density function. For each factor or asset i, $x_{i,j}$ is the jth simulated return scenario, μ_i is the target stress return, and σ_i is the sample volatility across scenarios. We assume uncorrelated target marginal distributions, which results in the removal of the term $c^p(\Phi_1^g(x_1), \ldots, \Phi_m^g(x_m)|\Theta_1)$ in Equation 3. α is a positive scaling parameter on the standard deviation and is set to be $W^{-0.5}$. We prefer $W \approx ESS$ and call the resulting α optimal because $\alpha\sigma$ can be subsequently viewed as the standard error that is consistent with ESS. After computing the likelihood weights, we calculate the weighted average returns of all factors and assets as they represent the conditional average returns under the specified stress market event.

To conduct the stress test, 50,000 asset return scenarios are generated by configuring the long-term risk model to simulate two-year returns instead of one. The two-year risk horizon allows the risk model to generate more extreme returns to respect the absolute continuity condition when our methodology is applied. Exhibit 4 reports the results for the four stress cases with the conditional average returns for each asset, factor, and the portfolio, alongside the effective sampling size, optimal α, and mean squared error between the target stress returns and conditional average returns. We use the hypothetical portfolio in Exhibit 1 and the m_1 factor definition. The first two cases involve stressing the factors, whereas the last two cases involve stressing two selected assets. A quick observation from Exhibit 4 shows that the target stress return and the subsequent weighted average return on the same factor or asset in the simulation are similar, as expected. This suggests that our methodology can successfully zoom in to a region within the simulated return distribution that centered on the desired target stress values.

The first case involves the equity, fixed income, and inflation factor shocks of –50%, 10%, and –30%, respectively. This scenario can be interpreted as a severe recession. As expected, most equity and inflation assets suffer large losses, leading

EXHIBIT 4

Portfolio Stress Testing

		Cases for Market Stress Scenarios with Target Stress Returns			
Assets	Portfolio Weight %	EQ F: –50% FI F: 10% INF F: –30%	EQ F: 20% INF F: 20%	EQUS: –40% FIUS10: 10%	GSCIEN: –60% GSCIPre: 20%
EQ-AU	3.2	–50.52%	21.03%	–41.24%	–1.25%
EQ-CA	3.2	–50.54%	20.22%	–41.43%	–3.41%
EQ-FR	3.2	–57.03%	20.92%	–42.02%	1.44%
EQ-DE	3.2	–60.57%	20.27%	–46.33%	–3.41%
EQ-UK	3.2	–42.35%	18.56%	–33.28%	1.90%
EQ-JP	4.0	–47.85%	19.61%	–32.77%	–3.40%
EQ-US	20.0	–44.36%	18.92%	**[–40.17%]**	2.00%
FI-AU	4.0	8.18%	–4.50%	1.03%	0.72%
FI-CA	4.0	9.32%	–5.32%	3.76%	0.83%
FI-FR	4.0	8.65%	–3.91%	2.82%	1.99%
FI-DE	4.0	10.60%	–3.29%	4.63%	1.25%
FI-UK	8.0	12.57%	–3.21%	5.42%	0.19%
FI-JP	4.0	4.17%	–3.34%	0.25%	–1.21%
FI-US10	12.0	15.63%	–6.74%	**[9.96%]**	2.88%
CORP-US	5.0	1.29%	–0.10%	2.23%	2.28%
COM-EN	5.0	–51.92%	34.90%	–24.33%	**[–59.75%]**
COM-ARG	5.0	–36.59%	6.19%	–17.38%	–17.47%
COM-PM	5.0	–4.93%	18.26%	–2.44%	**[19.92%]**
EQ F	$m_1 - F_1$	**[–50.46%]**	**[19.93%]**	–39.61%	–0.88%
FI F	$m_1 - F_2$	**[9.88%]**	–4.33%	3.98%	0.95%
INF F	$m_1 - F_3$	**[–31.35%]**	**[19.93%]**	–14.81%	–19.51%
Portfolio		*–19.23%*	*8.88%*	*–15.85%*	*–2.13%*
ESS		23	239	87	149
MSE		0.00687%	0.00004%	0.00015%	0.00034%
Optimal α		0.22	0.06	0.11	0.09

NOTES: Values in square brackets denote asset/factor scenarios that are used to compute the likelihood weights as per our methodology. Based on these weights, the weighted average (conditional) asset returns and the weighted average portfolio return are computed. The simulation consists of 50,000 scenarios over a two-year risk horizon. EQ F, FI F, and INF F denote the three factors (F_1, F_2, and F_3) defined in factor set m_1. EQUS, FIUS10, GSCIEN, and GSCIPre are defined in online Appendix Exhibit C1. We report the mean squared error (MSE) between the target stress returns and conditional average returns after reweighting as well as the effective sampling size (ESS) and the optimal α.

to a portfolio loss of 19.23%. The second case involves modest equity and inflation factor shocks of 20% each. This contributes to positive returns to equity and inflation assets, generating a portfolio gain of 8.88%. The negative fixed-income asset returns are understandable because a high inflation shock coupled with positive equity returns is likely disadvantageous for fixed-income assets. The third case involves S&P 500 and US 10-year Treasury shocks of –40% and 10%, respectively. This case could represent a US recession scenario. Our aim is to examine how this affects other foreign markets. Equities globally look to be severely affected, but foreign fixed incomes are less so. This may suggest that, with historical data, the impact of US monetary policy tightening or easing may not systematically affect foreign fixed income. The fourth and final case involves energy commodity and precious metal price shocks of –60% and 20%, respectively. These two-sided commodity shocks on inflationary assets create a dilemma because financial asset prices could rise or fall depending on the reasons for these inflation shocks. As a result, both the equity and fixed-income factors on average produce near zero conditional returns. In reality, the outcome for equity returns would likely be significantly positive or negative. If practitioners have a view on whether this is a supply-side or demand-side shock, they should impose their views of the anticipated equity shocks in our methodology to produce more meaningful results.

CONCLUSION

In this article, we address the potential misalignment created by models used for strategic portfolio construction and investment risk management due to differences in assumed asset characteristics. We propose a methodology inspired by change of measure and importance sampling techniques. The method can be applied to alter the simulated scenarios generated by an investment risk system calibrated with short-term history to ones that reflect longer-term asset characteristics. Our methodology thus enables better consistency between models used for investment and those used in risk management. Because of its ease of usability, the likelihood weighting scheme could motivate institutional managers to investigate the use of a single system for both investment and risk decision making. Attention could be shifted to improving the quality and range of scenarios generated from existing systems instead of aiming to implement new ones. Furthermore, we show that this framework has potential applications related to other types of portfolio analysis, such as sensitivity analysis and stress testing. Finally, given the rise of high-performance computing and processing capabilities, we argue that our method would be more effective and applicable for a larger set of investment and risk applications in the future.

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