Price Competition with Capacity Uncertainty - Feasting on Leftovers*

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Abstract

There is ample empirical evidence documenting that large firms set significantly lower prices than smaller, capacity-constrained, firms. This is paradoxical in light of the standard theoretical result that large firms charge higher prices than small firms in models of price competition with capacity constraints. We argue that private information about capacity constraints can account for this puzzle. We provide concavity conditions on the demand and on the type distribution under which there exists a unique, monotone decreasing price equilibrium. Solving the model requires a novel approach of studying several different regions of pricing incentives depending on the realized capacity levels. We show that firms with intermediate capacities compete in an auction type interaction, while firms with low or high capacity levels compete less vigorously on the margin.

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1 Introduction

Context. One of the standard results in the vast literature on price competition with capacity constraints is that larger firms tend to choose higher prices than smaller firms. This theoretical prediction has not been corroborated empirically. Instead, several empirical studies document the opposite behavior: firms with larger available capacities set significantly lower prices than smaller firms.[1]

In this paper, we show that private information about firms' available capacity levels is a possible explanation for this puzzle. The literature on Bertrand-Edgeworth competition typically assumes that capacity levels of firms are observable to all market participants. But consider a third-party online ticket seller that must choose at which price to sell its remaining seats for a concert. It will usually not know how many seats competing sellers have for the same concert. In practice, therefore, there are good reasons to assume that firms have a better understanding of their actual production capacity compared to their rivals. We highlight three specific reasons for this. First, producers often have privileged knowledge about the contracts they have signed with their suppliers. Even when capacity investments are common knowledge to all market participants, available capacity may be lower than the firms' nominal capacity. For instance, Hortaçsu and Puller (2008, p. 5) explain that this phenomenon plays a key role in electricity spot markets. Second, the possibility of production failures generates capacity uncertainty. Firms, for strategic reasons, may try to hide their "failure rate" from competitors, thus obfuscating their available production capacity. This obfuscation/revelation decision has also been discussed in the operations management literature in the context of quantity competition (Ye et al., 2013; Durango-Cohen and Wagman, 2014). Third, uncertainty in weather forecasts affects the maximally available output both for renewable electricity producers and for agricultural producers. In the agricultural sector, the actual annual yield of crops is uncertain and unobservable to outsiders. Using a model with Cournot competition, Farmer (1994) describes how Brazil's repeated decision not to reveal its annual harvest capacity contributed to sustaining the successful international coffee cartel between 1962 and 1989.

Importantly, Fabra and Llobet (2022), whose work we discuss below, provide empirical evidence that strongly suggests that firms have a much better under-

[1] De Silva et al. (2003); De Silva (2005); Escobari (2012); Fonseca and Normann (2013); Jofre-Bonet and Pesendorfer (2003); Melnick et al. (1992). We discuss these papers in more detail in the related literature section.

standing of their own capacity than that of other firms. Given this, we study the impact of private information about capacities on the firms' pricing strategies.

Model, Main Results and Monotone Incentives. We analyze a duopoly model of price competition with homogeneous goods in which the capacity of each firm is private information. The distribution of capacities is common knowledge, and symmetric across the two firms. Our first contribution is to address the empirical observations by studying general conditions on the primitives of the model that generate equilibria where prices are monotone decreasing in capacities. We show that a unique symmetric, monotone equilibrium exists when the type distribution of capacities, and the demand function are both concave (Proposition 2).[2]

Thus with private information, capacity-constrained firms prefer to charge a high price to whatever demand is left by the other firm. Intuitively, the capacity-constrained firm's only hope is to serve the eventual residual demand, i.e., it hopes to feast on leftovers. This result sharply contrasts with the full information case in which the low capacity firm charges a lower price in expectation. We also demonstrate, in Proposition 1, that monotonically *increasing* price equilibria do not exist because charging the highest (equilibrium) price yields the same leftover demand regardless of the type of the "losing" firm but charging any lower price allows a firm with a higher capacity to expand more than a lower capacity firm is able to. Therefore, in equilibrium the highest price has to be charged by the firm with the lowest capacity.

Our second contribution is to observe that without regularity assumptions on the demand and capacity distribution function, the monotonicity result does not always hold. Through a simple example with two types, we show that non-decreasing densities of types may lead to a violation of the single crossing condition, and thus to the non-existence of a monotone (decreasing) equilibrium. The example illustrates the different trade-offs faced by high and low capacity firm when changing their prices. On one hand, a high capacity firm faces a larger revenue loss when reducing its price (price effect of a price reduction). On the other hand, a firm with a higher capacity can expand its production easier to serve the extra demand when reducing its price (quantity effect). If the price effect dominates the quantity effect, the single-crossing condition fails, and no equilibrium with monotone decreasing prices exists. In the two-type example,

[2] Depending on the size of capacities, concavity of the demand function can be substantially relaxed.

we show that the single crossing condition may fail only if the more likely type is the type with the higher capacity level. Building on this insight, we show that imposing concavity on the distribution function in the general model is sufficient to maintain the single crossing condition, and thus to obtain a monotone decreasing equilibrium.

Pricing Incentives in a General Model with Capacity Constraints. Our model provides a general framework to analyze pricing under capacity constraints with incomplete information by studying general demand and distribution functions in an otherwise standard duopoly game. This generality in our modeling approach leads to a complex pattern in the strategic incentives of the firms in two ways. First, depending on the realized but privately observed capacity constraints, the firms may or may not compete vigorously in prices. When a firm has a capacity in the mid range it pays to become the firm with the lower price to be able to utilize capacities. In this range of capacities, the strategic situation resembles that of auctions. When capacities are so high that full utilization does not occur in equilibrium, firms become indifferent between a whole range of prices, and price competition is absent on the margin. Similarly, when capacities are very low, capacities are utilized even if the other firm charges the lower price. In this case, price competition is absent on the margin as well.[3] Our paper thus highlights that intensive price competition is most likely to occur when the realized capacities are neither too high nor too low, an insight useful for future applications. Second, as we highlighted above, strategic incentives may support or undermine the existence of a monotone equilibrium. Importantly, the single crossing conditions of global optimality take different forms in the low, medium and high ranges for capacities.

Our analysis needs to solve two potential problems related to those two points. To keep the analysis tractable, we need to prevent the different types of (marginal) pricing incentives from appearing in a non-monotone way as capacities change, and we also need to ensure that the single crossing conditions are maintained. Concavity of the demand and concavity of the distribution of capacity levels achieve both goals but depending on the range of possible capacities concavity of demand can be significantly relaxed.

The rest of the paper is organized as follows. Section 2 details the related literature. In Section 3.1, we present our modeling framework. In Section 3.2, we

[3] In this case, we identify a continuum of non-monotone equilibria that are payoff equivalent to the unique monotone equilibrium.

present our main results. Section 3.3 presents the formal analysis of the baseline case, whereas Section 3.4 presents a semi-formal discussion of two more cases. Section 4 provides a discussion and Section 5 concludes. Most of the proofs appear in the Appendix.

2 Related Literature

The conventional wisdom. Our paper contributes to the extensive literature on the Bertrand-Edgeworth models of capacity-constrained pricing. This literature has introduced uncertainty about different variables of the standard Bertrand-Edgeworth price competition model: cost uncertainty (Hartmann, 2006) and demand uncertainty (de Frutos and Fabra, 2011; Lepore, 2012).[4] To the best of our knowledge, ours is the first paper investigating classical Bertrand-Edgeworth competition under uncertain capacities. In particular, our model encompasses the seminal model of Levitan and Shubik (1972) as a special case in which both firms are (symmetrically) constrained. Another important connection to this literature is the puzzle we identify: all capacity-then-price models that can establish a clear relationship between the (mixed-strategy) equilibrium price distributions find that larger firms set stochastically higher prices. Kreps and Scheinkman (1983, p. 332); Gelman and Salop (1983, p. 319); Allen et al. (2000, p. 507) and Besanko and Doraszelski (2004, p. 28) all have this feature in models where there is no capacity uncertainty. Our main result, Proposition 2, states that this relationship between prices and capacities is easily reversed, under quite general conditions, in the presence of capacity uncertainty.

In particular, in the mixed strategy equilibria without capacity uncertainty, firms must choose prices from the same interval with the firm with the lower capacity never choosing the highest price, and the larger firm choosing this price with strictly positive probability.[5] In contrast, in our setting with capacity uncer-

[4]Demand uncertainty also plays a key role in the dynamic game of Boyer et al. (2012) where capacity building is followed by competition a la Cournot.

[5]As Kreps and Scheinkman (1983) write:

At first glance, it might be thought that firm 1, having the larger capacity, would profit more by underselling its rival, and therefore it would name the (stochastically) lower prices. But (as is usual with equilibrium logic) this is backwards: each firm randomizes in away that keeps the other firm indifferent among its strategies. Because firm 1 has the larger capacity, firm 2 is more "at risk" in terms of being undersold, and thus firm 1 must be "less aggressive."

Davidson and Deneckere (1986) further clarify:

When [the capacity of the small firm] $K_1 < D(0)$*, however, the large firm has an incentive to form a price umbrella under which the small firm can live. This is most easily seen when* K_1 *is very small: in that case the large firm may as well ignore the lower-priced competitor since* tainty, the Perfect Bayesian equilibrium is in pure strategies: every type charges a different price in equilibrium. Reasonable distributional assumptions on the type space (e.g. concavity) guarantee monotone incentives: firms with larger capacities charge lower prices.

The empirical puzzle. Our results are in line with most of the empirical evidence on the relationship between capacity constraints and prices which predominantly predicts that firms with higher available capacities charge lower prices in several industries. In the road construction industry, Jofre-Bonet and Pesendorfer (2003) show that larger firms tend to bid more aggressively in procurement auctions. Similarly, De Silva et al. (2003) and De Silva (2005) find that unconstrained firms bid more aggressively for road construction contracts than their capacity-constrained rivals. In the airline industry, Escobari (2012) documents that fewer available seats lead to higher prices. In the health care sector, Melnick et al. (1992) show that hospitals with lower excess capacities charge higher prices. In all of these industries, firms arguably only have limited knowledge about the available capacities of their rivals. Moreover, in a lab experiment, Fonseca and Normann (2013) show that lower production capacities lead to higher prices. The only exception we are aware of is the corn industry in which Ilin and Shi (2021) find that unconstrained firms charge higher prices.

More directly related to our approach, Fabra and Llobet (2022) make a strong case that private information about capacities matters in some important markets. They compare how actual hourly energy production levels of six Spanish energy plants depend on forecasts based on public information compared to privately known forecasts,[6] and show that the forecasts based on private information predict actual production levels significantly more accurately than the forecast based on public information only. Fabra and Llobet (2022) conclude that *"The evidence suggests that firms possess private information that allows them to significantly improve the precision of the forecasts of their own plants' available capacities."*

The literature with private capacities. Our model is closest to Fabra and Llobet (2022) who describe the market for renewable energy as a (uniform-price or discriminatory) auction between wind and solar producers with unobservable

he is of insufficient size. Of course, a pure-strategy equilibrium of this type will not exist. The mixed-strategy solution, on the other hand, does have this feature: the large firm has a mass point at the upper end of the support (the monopoly price) that becomes larger as K_1 decreases.

[6] Fabra and Llobet (2022) obtained proprietary, and hence private, forecasts from these six electricity operators. See Table 1 on page 31 of their paper.

capacity. In their model, firms submit a price-quantity pair to the auctioneer, specifying a minimal price at which they are willing to supply the specified quantity. The most interesting finding from our perspective is that in such auctions, equilibrium price offers are strictly decreasing in the unique symmetric equilibrium. The major difference between our work and Fabra and Llobet (2022) is that we study a canonical Bertrand-Edgeworth model with capacity constraints where firms choose their own prices, while they analyze a centralized market where the price is formed via a uniform price auction.[7] As a result the two approaches complement each other in applications. While equilibrium existence holds under standard assumptions on capacities when markets are centralized, the literature has not provided such conditions for our Bertrand-Edgeworth model. Our main theoretical contribution is to present general conditions under which a (unique) monotone decreasing equilibrium exists (or fails to exist) and to provide underlying economic intuition. We obtain the following additional insights. First, when demand is not perfectly inelastic, price monotonicity in equilibrium can easily break down. Second, we provide general conditions on *price sensitive* demand such that no symmetric, monotone increasing equilibrium exists, independent of the type space.

Our paper is also related to the literature on privately observed *inventory* selection. In their model, Montez and Schutz (2021) show that each store randomizes its price, ordering a low inventory when it sets a high price, and a high inventory when it holds a sale. The key distinction is that Montez and Schutz (2021) analyze industries with recurrent capacity building where capacities are very unpredictable (like retail), whereas our setting applies to industries in which baseline capacities are sluggish but *available* capacities are subject to some well-understood random fluctuations (e.g. manufacturing). Finally, our paper is also related to Janssen et al. (2023) who study the dynamics of competition under capacity uncertainty. An important difference with our model is that firms can only be of two types in Janssen et al. (2023), constrained or unconstrained, whereas we allow for a rather general family of capacity distributions. The two-period pricing problem in their model has a continuum of equilibria, some of them exhibiting monotonicity.

[7] Except for the case of inelastic demand where they also study discriminatory auctions. In this case, our baseline model encompasses the discriminatory auction case of Fabra and Llobet (2022). We discuss this relationship in more detail in Section 4 of our paper.

3 Duopoly price competition with privately observed capacity constraints

3.1 Setup and preliminary analysis

There are two firms with capacity k_i, k_j distributed independently and identically on $[\underline{k}, \overline{k}]$ according to an atomless distribution function F with a corresponding probability density function f. The capacities are private information to each firm, and costs are normalized to zero. The demand $D(p)$ is such that the revenue function $D(p)p$ is strictly concave, and $\overline{k} < D(0)$ holds, so no firm can serve the entire market by itself. The firm with the lower price makes the sales first (efficient rationing). This means that if $p_2 > p_1$, then the residual demand for firm 2 is

$$D_2 = \max\{0, D(p_2) - k_1\}.$$

We allow for any tie-breaking rule in the event that $p_1 = p_2$: in our equilibrium the form of the tie breaking rule is inessential as ties occur with zero probability in a monotone equilibrium.

We are looking for a symmetric monotone decreasing equilibrium in the game where firms set their prices. Let $\rho : [\underline{k}, \overline{k}] \to \mathbb{R}_0^+$ be a strictly decreasing equilibrium price strategy. First, we obtain the expected profit of a type k firm who chooses price $\rho(\widehat{k})$ as follows:

$$\widetilde{\pi}(k, \widehat{k}) = F(\max\{\widehat{k}, D(\rho(\widehat{k})) - k\})\rho(\widehat{k}) \min\{D(\rho(\widehat{k})), k\} + \int_{\max\{\widehat{k}, D(\rho(\widehat{k}))-k\}}^{\overline{k}} f(x) \max\{0, D(\rho(\widehat{k})) - x)\} dx \rho(\widehat{k}). \tag{3.1}$$

To explain (3.1), let us consider three cases.

Case 1. If $D(\rho(\widehat{k})) \in [k, k + \widehat{k}]$, then upon charging the lower price the firm utilizes its entire capacity k. This happens with probability $F(\widehat{k})$. If the firm charges the higher price, then it can serve the residual demand $\max\{0, D(\rho(\widehat{k}))- x)\}$.

Case 2. If $D(\rho(\widehat{k})) < k$, then upon charging the lower price the firm does not utilize its entire capacity and serves the entire market by selling $D(\rho(\widehat{k}))$. This happens with probability $F(\widehat{k})$. If the firm charges the higher price, then it can serve the residual demand $\max\{0, D(\rho(\widehat{k})) - x)\}$. For a small deviation with $\widehat{k}$ being close to k, this residual demand is zero.

Case 3. If $D(\rho(\widehat{k})) > k + \widehat{k}$, then upon charging the lower price the firm utilizes its entire capacity k. Moreover, the firm fully utilizes its entire capacity as long as the competitor's capacity $\widetilde{k}$ is such that $\widetilde{k} \leq D(\rho(\widehat{k})) - k$. When $\widetilde{k} > D(\rho(\widehat{k})) - k$ then the firm serves the leftover demand.

When analyzing small deviations to derive the first-order conditions of the firms' problems, we need to assume that $\widehat{k}$ is close to k. Then the above three cases simplify to whether $D(\rho(k)) \in (k, 2k)$, $D(\rho(k)) > 2k$ or $D(\rho(k)) < k$. In principle, there may be several different intervals of k where $D(\rho(k))$ belongs to those three different regions, which would lead to a complex, and ever changing pattern of first-order conditions as k changes. However, based on Example 1 (see below), we identify key concavity conditions[8] on D and F that are amenable to the existence of a monotone equilibrium *and* (as we show later) also guarantee that the patterns of $D(\rho)$ versus k and $2k$ are tractable. In particular, under those concavity conditions, we obtain that there could be at most three regions: i) $D(\rho) > 2k$ for small values of k; ii) $D(\rho) \in (k, 2k)$ for intermediate values of k; and iii) $D(\rho) < k$ for large values of k.

When characterizing a monotone equilibrium we work with an ordinary differential equation of ρ in k as determined by the first order condition, and we also use an initial condition on $D(\rho(\underline{k}))$. We establish in the Appendix that depending on model primitives, a monotone equilibrium may or may not have the first and the third regions but it always has the second region where $D(\rho) \in (k, 2k)$. The intuition is clear: when $D(\rho) < k$, capacities are not used fully in the margin, and thus incentives to price according to a monotone equilibrium are absent. When $D(\rho) > 2k$ the two firms are not competing on the margin because even the firm with the higher price can utilize its capacity fully. To summarize, there is no incentive for monotone pricing unless it holds for at least some k that $D(\rho) \in (k, 2k)$.[9]

The proof (presented in the Appendix) covers several cases: There are three

[8]The concavity of the demand (or for some results the concavity of the revenue function $D(p)p$) is a common assumption in the literature on oligopolistic games.

[9]The fact that equilibria are qualitatively different for low, medium, and high levels of capacity constraints is reminiscent of a standard finding in the literature of Bertrand-Edgeworth duopolies. In many models without uncertainty on capacities, only mixed-strategy equilibria exist for intermediate capacities whereas pure-strategy equilibria exist for low and high capacity levels (see e.g. Levitan and Shubik (1972); Kreps and Scheinkman (1983); Acemoglu et al. (2009)).

possible variations depending on how $D(\rho)$ compares to $2k$ and k, and there are two cases depending on how $D(\rho)$ compares to $\overline{k}$. That would give six cases but the case where $D(\rho) < k$ and $D(\rho) > \overline{k}$ cannot occur, so we have five cases overall. Fortunately, under concavity all of these cases can be shown to support a monotone equilibrium, and the equilibrium can be assembled by combining the different forms of ρ for the different regions of capacities.

In the simplest case where $D(\rho(\underline{k})) \leq 2\underline{k}$, and $D(\rho(\underline{k})) \geq \overline{k}$ the first order condition of the firms determines the same differential equation for all values of k (see expression (3.6)), and the analysis is more transparent. This is the only case we formally analyze in the main text, in Section 3.3. The other cases are formally analyzed in the Appendix, while two of those four other cases are given a semi-formal discussion in Section 3.4.

3.2 The General Result

We now present our two main results: 1. Regardless of the type space, there does not exist a symmetric, monotone increasing equilibrium, 2. If the demand and the type space are concave then there exists a unique monotone decreasing equilibrium.

We first show (Proposition 1) that the type of equilibrium that is found in the literature with known capacities in which high capacity firms tend to choose *higher* prices does not exist in our model with incomplete information.

Proposition 1. *If $D(p)p$ is strictly concave, there does not exist a symmetric, monotone increasing equilibrium.*

Proof. In such a monotone *increasing* equilibrium, type $\overline{k}$ would charge the highest price in equilibrium, and would sell a quantity of $\min\{D(\rho(\overline{k})) - k, \overline{k}\}$ where k is the capacity of the other firm. The lowest capacity type would set price $\rho(\underline{k})$, and sell quantity $\underline{k}$. On the other hand, by charging price $\rho(\overline{k})$, the low capacity type would sell quantity $\min\{D(\rho(\overline{k})) - k, \underline{k}\}$, while charging $\rho(\underline{k})$ would allow type $\overline{k}$ to sell a quantity that is larger than $\underline{k}$. It is clear that

$$\frac{\min\{D(\rho(\overline{k})) - k, \overline{k}\}}{\min\{D(\rho(\overline{k})) - k, \underline{k}\}} \leq \frac{\overline{k}}{\underline{k}}$$

for all k, and this inequality holds strictly for some k close to $\overline{k}$ because $D(\rho(\overline{k})) < 2\overline{k}$ otherwise no capacity constraint would ever bind in equilibrium, which would then induce the firms to increase their prices.

Therefore, the expectation of these ratios (over the other firm's type k) satisfies

$$\frac{E_k(\min\{D(\rho(\overline{k})) - k, \overline{k}\})}{E_k(\min\{D(\rho(\overline{k})) - k, \underline{k}\})}) < \frac{\overline{k}}{\underline{k}}. \tag{3.2}$$

Consequently, charging the price $\rho(\overline{k})$ allows both types to be able to sell more similar quantities, while charging $\rho(\underline{k})$ allows the type $\overline{k}$ to have much higher expected sales and profits than type $\underline{k}$ (a ratio of $\overline{k}/\underline{k}$). Therefore, (3.2) implies type $\overline{k}$ has more incentive to set $\rho(\underline{k})$ vs $\rho(\overline{k})$ when compared to the incentives of type $\underline{k}$. This implies that the incentive condition (single-crossing condition) for these two types are violated, and such an increasing equilibrium cannot occur. Q.E.D.

The proof illustrates that the basic incentive conditions require that the highest possible capacity type charges a lower price than the lowest possible capacity type, regardless of the type distribution F. This is a very general result since we did not assume that F was concave, and thus a monotone decreasing equilibrium may not exist.

The main lesson to be drawn is that in the classic Bertrand-Edgeworth model with privately known capacities prices cannot increase with capacity, contrary to the model in which capacities are publicly known. Furthermore, the empirical puzzle is in line with Proposition 1: it suggests that firms with larger capacities tend to set lower prices. We now illustrate that our model generates such price setting behavior in equilibrium, which is summarized in Proposition 2.

Proposition 2. *If F and D are both concave, then there is a unique symmetric monotone decreasing equilibrium.*

While the complete proof of Proposition 2 is detailed in the Appendix, we provide its main flavor in subsection 3.3 below in which we present the proof for the "baseline case" where $D(\rho) \in (k, 2k)$ for all values of k and $D(\rho(k)) > \overline{k}$.

Proposition 2 imposes some restrictions on the primitives of our model. In particular, it adds concavity of the type distribution as a sufficient condition. To see how the lack of concavity may fail to generate a monotone decreasing equilibrium we present a two-type example that can be perturbed to one in which the single crossing condition fails without regularity assumptions on the distribution function F.

Example 1. *Let $D(p) = 1 - p$, and assume that firm 1 has capacity $k_1 = 0.75$, and firm 2 has capacity $k_2 = 0.5$. The equilibrium can be characterized following the literature as follows. Both firms charge prices on interval $[1/12, 1/4]$ with distribution functions $F_1 = \frac{0.5p-1/24}{p(p+0.25)}$, and $F_2 = 3F_1/2$. Note, that the firm with the higher capacity places an atom of $1/3$ on price $p = 1/4$. Writing the indifference condition of firm 2, it holds for all $p \in [1/12, 1/4]$ that*

$$0.5p(1 - F_1(p)) + (1 - p - k_1)pF_1(p) = \pi^*$$

where π^* is the equilibrium profit of firm 2.

Let us now consider the incentives of a dummy type of firm 2 by introducing a zero probability type k not equal to 0.5. Then taking the equilibrium from above, the profit of dummy type k of firm 2 from charging price p is $p(1 - F_1(p))k + pF_1(p)(1-p-k_1) = \pi^* + p(1-F_1(p))(k-0.5)$. In our example, it is easy to show that $p(1 - F_1(p))$ is convex. This implies that $p(1 - F_1(p))(k - 0.5)$ is convex in p, and is maximized at $p = 1/12$ when $k > 0.5$. In this case, the monotonicity of firm 2's incentive holds because a higher capacity yields to an optimal choice of a lower price. However, when $k < 0.5$, $p(1 - F_1(p))(k - 0.5)$ is concave, and the new type's profits are maximized at an interior price $p \in (1/12, 1/4)$. This contradicts monotonicity of incentives because a lower capacity does not lead to choosing the highest price $p = 1/4$ but instead leads to choosing an intermediate price.

Example 1 teaches us the following insights:

1. If one perturbed this model by introducing a second type with a capacity below 0.5 with probability ε close to zero, then a monotone equilibrium would not exist in the perturbed model. With such a perturbation the actual equilibrium would include the two types of firm 2 mixing on overlapping intervals, which means that the equilibrium would be non-monotone. In the online Appendix 1, we work out a two-type example where such overlapping mixing occurs in equilibrium.

2. If the perturbation included a second type slightly above the most likely type, then a monotone equilibrium would still exist. This suggests that concavity of the distribution function where higher types are less likely is useful for the existence of a monotone equilibrium.

This example strongly suggests that concavity of the distribution function

F is an important condition to make sure that a firm with a lower capacity has the incentive to charge a higher price than a firm with a higher capacity. This key single-crossing condition is analyzed further in the proof of equilibrium existence below. In the baseline case analyzed in the Section 3.3, we show that the single-crossing condition holds when F is concave under the maintained regularity assumption that the revenue function is concave - see Proposition 3. However, in the other cases, when capacities can be either very high or very low, the assumption on demand has to be strengthened. Assuming a concave demand function is sufficient for the single-crossing condition (and thus for equilibrium existence) in all of the relevant cases. In our interpretation, the necessary condition on the demand function is fairly mild and the stronger assumption that delivers existence is the concavity of the distribution function.

3.3 Analysis of the baseline case

To gain intuition into the conditions of Proposition 2 under which a unique symmetric monotone decreasing equilibrium exists we focus the analysis on the case where $D(\rho) \in (k, 2k)$ for all values of k and provide sufficient conditions for the existence and uniqueness of a symmetric monotone equilibrium in this case. To further simplify exposition, we strengthen the condition that $D(\rho(k)) > k$ for all k, and assume that

$$D(\rho(k)) > \overline{k} \tag{3.3}$$

for all k.

Given (3.3) and $D(\rho(k)) < 2k$, one can simplify the payoff function as

$$\widetilde{\pi}(k, \widehat{k}) = F(\widehat{k})\rho(\widehat{k})k + \int_{\widehat{k}}^{\overline{k}} f(x)(D(\rho(\widehat{k})) - x)dx\rho(\widehat{k}). \tag{3.4}$$

We obtain the first order condition by setting $\partial\widetilde{\pi}/\partial\widehat{k} = 0$ at $\widehat{k} = k$. As a short hand notation, we introduce the equilibrium profits of each type as $\pi(k) = \widetilde{\pi}(k, k)$. The first order condition becomes

$$\rho'(k)\frac{\pi(k)}{\rho(k)} + f(k)\rho(k)k - f(k)(D(\rho(k)) - k)\rho(k) + D'(\rho(k))\rho'(k)(1 - F(k))\rho(k) = 0.$$

Let $\tau(k) = E(\widetilde{k} \mid \widetilde{k} \in (k, \overline{k}]) = \frac{1}{1-F(k)} \int_k^{\overline{k}} f(x) x dx$. We can rewrite the first order condition as

$$\rho'(k)[F(k)k + (1-F(k))(D(\rho) + D'(\rho)\rho - \tau(k))] = f(k)(D(\rho(k)) - 2k)\rho(k). \quad (3.5)$$

Upon using (3.5), (3.1), and rearranging we obtain that

$$\rho' = \frac{(D(\rho) - 2k) f(k) \rho(k)}{F(k)k + (1 - F(k))[D'(\rho)\rho + D(\rho) - \tau(k)]}. \quad (3.6)$$

To complete this differential equation with a boundary condition we need to characterize $\rho(\underline{k})$. In line with the literature on oligopoly games with capacity constraints, we argue that $\rho(\underline{k}) = p^* = \arg\max_p \{p(D(p) - E(k))\}$ where $E(k) = \int_{\underline{k}}^{\overline{k}} f(x) x dx = \tau(\underline{k})$ is the expected capacity of a player.[10] Such a price p^* is well defined (unique) because the revenue function $pD(p)$ is strictly concave. To see why $\rho(\underline{k}) = p^*$, note first that the profit of a firm when setting a price $p \geq \rho(\underline{k})$ is $p(D(p) - E(k))$ because in this case the firm only obtains the residual demand in equilibrium as his opponents sets a lower price than p with probability 1. Therefore $\rho(\underline{k}) \geq p^*$ must hold otherwise setting p^* would be a profitable deviation for a firm with a capacity $\underline{k}$. Also, $\rho(\underline{k}) \leq p^*$ because deviating to a slightly lower price than $\rho(\underline{k})$ yields a profit of at least $p(D(p) - E(k))$, which then implies the result.

The condition (3.5) implies that each seller maximizes his payoffs locally, that is for prices close to his equilibrium price $\rho(k)$. As standard in Bayesian games with monotone strategies, it is sufficient for global optimality if (3.5) holds, together with the single crossing condition $\partial \widetilde{\pi}^2 / \partial k \partial \widehat{k} > 0$, which simplifies to $(F\rho)' > 0$ by (3.4). The single crossing condition $(F\rho)' > 0$ is equivalent to $\rho' > -\rho f / F$. Using expression (3.6), the single crossing condition is equivalent to

$$\frac{(D(\rho) - 2k) f(k) \rho(k)}{F(k)k + (1 - F(k))[D'(\rho)\rho + D(\rho) - \tau(k)]} > -\frac{\rho(k) f(k)}{F(k)}.$$

Since the denominator of the left hand side is positive, this condition is equivalent to $F(k)(D(\rho) - 2k) > -F(k)k + (1 - F(k))[-D'(\rho)\rho - D(\rho) + \tau(k)]$, which can

[10]Further standard assumptions on D will guarantee that the arg max is unique.

be rearranged as

$$F(k)(D(\rho)-k)+(1-F(k))[D'(\rho)\rho+D(\rho)-\tau(k)]>0. \tag{3.7}$$

Therefore, the single crossing condition is equivalent to (3.7).

We have established that if there is a solution to the differential equation (3.5) that satisfies the relevant initial value condition $\rho(\underline{k})=p^*$, and also $(F\rho)'>0$ or equivalently (3.7) for every k, and $D(\rho)\leq 2k$ for all k, then this solution is a monotone and symmetric equilibrium of the game. Note, that under $D(\rho)<2k$ the single crossing condition (3.7) implies $\rho'<0$ automatically via (3.6) so the equilibrium will be decreasing. Also, $F\rho$ is strictly increasing at $k=\underline{k}$ because $F(\underline{k})=0$ so the single crossing condition holds at $\underline{k}$. Assume that $D(\rho(\underline{k}))=D(p^*)<2\underline{k}$. Then for small values of $k\in(\underline{k},\underline{k}+\varepsilon)$ it holds that $\rho'(k)<0$, and $D(\rho(k))<2k$. Suppose that for some greater k, $D(\rho(k))=2k$. At that point, $\rho'=0$ because the denominator of (3.6) is equal to $F(k)k+(1-F(k))[D'(\rho)\rho+D(\rho)-\tau(k)]=F(k)(D(\rho)-k)+(1-F(k))[D'(\rho)\rho+D(\rho)-\tau(k)]>0$, and the numerator is zero. But that would imply a contradiction because if $D(\rho(k-\varepsilon))<2(k+\varepsilon)$ for all $\varepsilon>0$, and $\rho'(k)=0$, then $D(\rho(k))=2k$ cannot hold. Therefore, we have the following result:

Lemma 1. *Suppose that $D(p^*)<2\underline{k}$, and $D(p^*)>\overline{k}$. Then if the solution ρ_1 of the initial value problem consisting of $\rho(\underline{k})=p^*$, and (3.6) satisfies (3.7), then ρ_1 forms a symmetric, and monotone decreasing price equilibrium.*

In the Appendix, Lemma 7 shows that a solution ρ_1 exists for the initial value consisting of $\rho(\underline{k})=p^*$ and (3.6), so we only need to inspect whether condition (3.7) holds for the solution ρ_1.[11]

It is useful to briefly review the various steps in our reasoning. We started by assuming that $D(p^*)<2\underline{k}$, $D(p^*)>\overline{k}$, F is concave and the revenue function $D(p)p$ is strictly concave and obtained, in a first step, that expressions (3.6) and (3.7) as necessary conditions for any equilibrium. In the next step, with the uniqueness and the existence result, we show that the initial value problem defined in the first step has a unique solution where $D(\rho)<2k$ holds for all k after inspecting how the differential equation works. This is our candidate equilibrium. In the last step we also prove that the solution to (3.6) is monotone and satisfies

[11] In Lemma 7 in the Appendix, we prove existence and uniqueness of the relevant solutions for the five cases of the model.

the single crossing condition (3.7). Putting these three steps together implies that the candidate equilibrium is indeed the unique symmetric and monotone (decreasing) equilibrium.

Proposition 3. *Assume that $D(p^*) < 2\underline{k}$, $D(p^*) > \overline{k}$, F is concave and the revenue function $D(p)p$ is strictly concave. Then there is a unique symmetric, monotone (decreasing) equilibrium that solves the initial value problem consisting of (3.6) and $\rho(\underline{k}) = p^*$.*

Proof. Lemma 7 in the Appendix implies that the initial value problem has a unique solution. Given Lemma 1, it is then sufficient to prove that the solution of the stated initial value problem ρ_1 satisfies (3.7). Let $\delta(k) = F(k)(D(\rho_1) - k) + (1 - F(k))[D'(\rho_1)\rho_1 + D(\rho_1) - \tau(k)]$ be a short notation for the left hand side of (3.7). We need to show $\delta(k) \geq 0$ for all k. Since $\delta(\underline{k}) = 0$ by construction, it is sufficient to show that for any k, $\delta(k) = 0 \Longrightarrow \delta'(k) \geq 0$. Taking the derivative, and letting the revenue function $R(p) = D(p)p$, we obtain

$$\delta'(k) = f(k)(D(\rho_1) - k - D'(\rho_1)\rho_1 - D(\rho_1) + \tau(k)) +$$

$$+F(k)(D'(\rho_1)\rho_1' - 1) + (1 - F(k))(R''(\rho_1)\rho_1' - \tau') >$$

$$f(k)(D(\rho_1) - k - D'(\rho_1)\rho_1 - D(\rho_1) + \tau(k)) +$$

$$+F(k)(D'(\rho_1)\rho_1' - 1) - f(k)(\tau - k).$$

using $R'' < 0$, and $(1 - F(k))\tau' = f(k)(\tau - k)$. Substituting in $\delta(k) = 0$ now yields $-[D'(\rho_1)\rho_1 + D(\rho_1) - \tau(k)] = \frac{F(k)(D(\rho_1)-k)}{1-F(k)}$. By the definition of $\delta(k)$ and via (3.7), $\delta(k) = 0$ implies $(\rho(k)F(k))' = 0$, so it also holds that $\rho_1' = -\rho_1 f/F$. Then it can be further written that

$$\delta'(k) > f(k)(D(\rho_1) - k + \frac{F(k)(D(\rho_1) - k)}{1 - F(k)}) - D'(\rho_1)\rho_1 f(k) - F(k) - f(k)(\tau - k) =$$

$$= f(k)(D(\rho_1) - D'(\rho_1)\rho_1 - \tau) + F(k)(f(k)\frac{D(\rho_1) - k}{1 - F(k)} - 1).$$

Using that $D(\rho_1) > D(p^*) \geq \overline{k} \geq \tau$ we have that $\delta' > F(k)(f(k)\frac{D(\rho_1)-k}{1-F(k)} - 1)$. Concavity of F implies $f(k) \geq \frac{F(\overline{k})-F(k)}{\overline{k}-k} = \frac{1-F(k)}{\overline{k}-k}$, so we have $f(k)\frac{D(\rho_1)-k}{1-F(k)} - 1 \geq \frac{D(\rho_1)-k}{\overline{k}-k} - 1 = \frac{D(\rho_1)-\overline{k}}{\overline{k}-k} > 0$, which concludes the proof. Q.E.D.

Compared with our general result in Proposition 2, the only difference in the necessary conditions in the general case is that D is concave, which is stronger than strict concavity of the revenue function, which is a fairly standard, and mild condition. While the result for the general case is not very different from the baseline case, the proof is more complex because it needs to address the other cases depending on how $D(\rho)$ compares with k and $2k$ as discussed in Section 3.1.

3.4 Very small and very large capacities

To gain further insight, we present a semi-formal discussion in which we highlight two further cases: when capacities are very small or very large.

3.4.1 The case of very small capacities

First, let us assume that capacities may be so low that $D(p^*) > 2\underline{k}$ holds, and thus the form of the profit function changes for k close to $\underline{k}$. Maintaining the assumption that $D(\rho(\widehat{k})) > \overline{k}$ but assuming now that $D(\rho(k)) > 2k$, we obtain (for $\widehat{k}$ close to k) that

$$\widetilde{\pi}(k, \widehat{k}) = F(D(\rho(\widehat{k})) - k)\rho(\widehat{k})k + \int_{D(\rho(\widehat{k}))-k}^{\overline{k}} f(x)(D(\rho(\widehat{k})) - x)dx\rho(\widehat{k}).$$

The first order condition becomes $\rho'(k)\frac{\pi(k)}{\rho(k)} + D'(\rho)\rho'\rho(1 - F(D(\rho) - k))) = 0$ or $\frac{\pi(k)}{\rho(k)} + D'(\rho)\rho(1 - F(D(\rho) - k))) = 0$. Upon substitution, $F(D(\rho) - k)k + \int_{D(\rho)-k}^{\overline{k}} f(x)(D(\rho) - x)dx + D'(\rho)\rho(1 - F(D(\rho) - k))) = 0$. Defining $\gamma(k) = D(\rho) - k$, and rearranging, this can be rewritten to obtain the following first order condition:

$$F(\gamma)k + (1 - F(\gamma))[D(\rho) + D'(\rho)\rho - \tau(\gamma)] = 0. \tag{3.8}$$

We can show that under our concavity assumptions, ρ is a solution of an initial value problem.

This first order condition is intuitive: each seller's problem can be decomposed into two regions. When the opponent has a type less than γ, then a price increase yields extra revenues on all of its capacity, k units. When the opponent has type greater than γ, then a price increase changes revenues in the usual manner for monopolies on a residual demand curve. In particular, there is a price effect that increases revenues on $D(\rho)-\tau(\gamma)$ units but some demand is lost when the price is increased (hence the $\rho D'(p)$ term). Any competitive advantage from being the firm with the lower price is absent because demand changes continuously for firm k when it faces an opponent whose type is in the neighborhood of γ. For this reason the first order condition does not contain the term ρ', and the interaction does not resemble an auction in this region of capacity types.[12]

Next, we show that the low capacity assumption cannot hold for all capacity levels (Lemma 2):

Lemma 2. *The assumption that $D(\rho) > 2k$ cannot hold for higher values of k when ρ is defined by (3.8).*

Proof. Note, that (3.8) cannot hold if $F(\gamma)$ is close to 1, which implies that for k large enough $\gamma < k$, which implies that $D(\rho(\overline{k})) < 2\overline{k}$. Q.E.D.

In view of Lemma 2, there must exist $k^* \in (\underline{k}, \overline{k})$ such that $D(\rho(k^*)) = 2k^*$, and for all $k < k^*$ it holds that $D(\rho) > 2k$ while for all $k > k^*$ it holds that $D(\rho) < 2k$. Formally, let ρ_2 be defined by (3.8), and let $k^* = \min\{k \mid D(\rho(k)) = 2k\}$. Then we can explicitly define a candidate equilibrium such for each $k \leq k^*$, the pricing function is equal to ρ_2, and for each $k > k^*$ the pricing function solves (3.6) with initial condition $\rho(k^*) = \rho_2(k^*) = D^{-1}(2k^*)$.

In Appendix 1 we show that, under our concavity assumptions, the solution to the first order condition (3.8) is strictly decreasing and satisfies the single crossing condition. This then implies the following result:

Proposition 4. *Assume that the type distribution F is concave, and the revenue function is concave. If $D(p^*) > 2\underline{k}$, and $D(\rho_2(\underline{k})) \geq \overline{k}$, then the unique symmetric, and monotone equilibrium consists of $\rho = \rho_2$ satisfying (3.8) for $k < k^*$, and $\rho = \rho_1$ satisfying (3.6) with initial condition $\rho_1(k^*) = 2k^*$ for $k \geq k^*$.*

Proof. The unique solutions ρ_1, ρ_2 and k^* are well defined by Lemma 7. Moreover, $\rho_1', \rho_2' < 0$, and global optimality conditions are satisfied under these assumptions as shown above and in the Appendix (for case 2). Therefore, a unique monotone equilibrium of the required form exists. Q.E.D.

[12] Another way to see this is to note that a higher capacity is not utilized in the margin when $D(p) > 2k$, and thus k does not affect profits. But then one can divide through by ρ' in the first order condition.

3.4.2 The case of very large capacities

Both our baseline case, and Proposition 4 have assumed that capacities are relatively small, and thus $D(\rho(\underline{k})) \geq \overline{k}$ holds. To provide further intuition, let us consider now the polar opposite case where for some k it holds that $D(\rho(k)) < k$ in the candidate monotone equilibrium. In the Appendix, we show that when D is concave the candidate monotone equilibrium (as characterized by an appropriately modified first order condition) is still an equilibrium. But for a type k such that $D(\rho(k)) < k$, the firm's capacity does not bind in equilibrium. Therefore, a firm with a slightly higher or lower capacity has the same decision problem to consider (for small price changes). So, it is not surprising that non-monotone equilibria exist in this case. However, all the non-monotone equilibria that we discover provide the same payoffs to all types as the monotone equilibrium:

Proposition 5. *If the unique symmetric, monotone decreasing equilibrium exhibits $D(\rho(k)) < k$ for some k, then there exist a continuum of non-monotone equilibria that are pay-off equivalent for all types to the underlying monotone equilibrium.*

The key optimality condition for the range of capacities where $D(\rho(k)) < k$ takes the form of indifference when considering a slight change in the price. In this region of k, setting the higher price yields zero sales, and setting the lower price $\rho(\widehat{k})$ yields sales of $D(\rho(\widehat{k}))$. Total profits then equal to $D(\rho(\widehat{k}))\rho(\widehat{k})F(\widehat{k})$, which needs to be constant in the chosen strategy $\widehat{k}$. Given the exogenous D, F functions the fact that $D(\rho)\rho F$ is constant pins down function ρ. Moreover, the single crossing condition holds as an indifference condition. The function ρ given by this indifference condition is strictly increasing under strictly concave revenues because $D(p)p$ is decreasing in the relevant range of prices where the price is lower than the monopoly price. However, we need to make sure that the induced ρ satisfies $D(\rho) < k$. This is ensured by our concavity assumptions on D and F.[13] Finally, note that we have not proved that non-monotone equilibria (or asymmetric monotone equilibria) do not coexist with the monotone equilibrium when $D(\rho(k)) > k$ in the monotone equilibrium but it is our conjecture based on insights from the literature on auctions.

[13]Concavity of demand can be somewhat relaxed for ρ to be still monotone. However, it is shown in the Appendix that (strict) concavity of the *revenue* function is not sufficient in general for ρ to be monotone, and thus for a monotone equilibrium to exist.

4 Discussion: Connection of our results to Fabra and Llobet (2022)

In this section, we explore more in detail the relationship and differences between our model and that of Fabra and Llobet (2022).

One key difference between our model and the discriminatory auction model of Fabra and Llobet (2022) is that the latter assumes a perfectly price-inelastic demand whereas we assume price-elastic demand. The discriminatory auction case of Fabra and Llobet (2022) is a limit case of our baseline model when D' tends to zero. Our baseline model thus encompasses the discriminatory auction case of Fabra and Llobet (2022) with zero marginal costs.

To see this, observe that by a simple transformation[14] of equation (3) part (i) of Fabra and Llobet (2022, p.13) describing the discriminatory auction case, one gets (using our notation):

$$-\frac{\rho'^{*}\left(k_i\right)}{\rho^{*}\left(k_i\right)-c_i}=\frac{\left(2k_i-D(\rho)\right)f\left(k_i\right)}{k_iF(k_i)+\left(D(\rho)-E\left(k_j \mid k_j\leq k_i\right)\right)\left(1-F\left(k_i\right)\right)}$$

Setting the marginal cost $c_i=0$, and recalling that we denote $E\left(k_j \mid k_j \leq k_i\right)=\tau(k_i)$, the above equation can clearly be obtained as a special case of our equation (3.6) with price-inelastic demand ($D'=0$).

To demonstrate the importance of elastic demand for the existence of monotone equilibria, we consider an extension of our Example 1 contained in online Appendix 2. In particular, let $D=1-ap$ where $a=0$ nests the inelastic demand case of Fabra and Llobet (2022). In the inelastic case, Fabra and Llobet (2022) show that in a discriminatory auction there exists a decreasing price equilibrium independent of the type distribution F. In online Appendix 2, we show that when $a>0.15$ a decreasing equilibrium does not exist.

5 Conclusion

To address the empirical observation that firms with higher capacities tend to price more aggressively, an observation that has not been obtained in formal models of oligopolies with general demand functions, we provide a novel model of price setting with capacity constraints under incomplete information. The

[14] Indeed, one can use the same transformation as Fabra and Llobet (2022, p.10) use to obtain their equation (2).

private information assumption on capacities is also relevant in many industries including airlines, hospitals, electricity markets and procurement. In our model, we find a unique monotone equilibrium where firms with higher capacities charge lower prices. We provide sufficient conditions for such an equilibrium to exist with conditions that are standard, relatively mild, and are easy to interpret.

In the course of proving our results we need to employ a piecemeal method to reflect the different incentives firms may have to choose their prices depending on how severe capacity constraints are. In effect, we provide a unified framework that allows one to consider severe capacity constraints in the same framework as very mild capacity constraints. This modeling approach may be particularly useful when it is hard to predict whether *realized* capacities ex-post may or may not be binding for the different firms.

Our main theoretical contribution is the discovery of simple concavity assumptions, and the explanation of why concavity is amenable to monotone decreasing pricing incentives. A further interesting aspect of our model is that we identify cases where potentially non-binding capacity constraints may lead to a continuum of payoff-equivalent equilibria. This last result points to an observation that applies to other models of capacity constraints: in the completely realistic case that *some* types of a firm may not be constrained by its capacity at all, we show that equilibrium multiplicity and indifference between charging different prices may well obtain in equilibrium.

Finally, it would be interesting to understand whether private information about capacities softens or intensifies competition. It is known that for the classic Bertrand model with constant marginal costs, introducing cost uncertainty may intensify price competition; see Lagerlöf (2023). Whether a simular result holds in our setting with uncertain capacities is a question we leave for future research.

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6 Appendix - Proofs

6.1 Proof of Proposition 2

In this appendix, we prove our main result, Proposition 2. We start by previewing all the cases and the notation for the different pricing functions that apply to the different cases in the candidate equilibrium as determined by the first-order conditions. As we will see, the analysis for many of the cases can be derived as simplifications of more complicated cases. The following table provides the notation for the pricing functions in the five different cases:

case	constraint 1	constraint 2	notation for ρ
1	$D(\rho) > \overline{k}$	$D(\rho) \in (k, 2k)$	ρ_1
2	$D(\rho) > \overline{k}$	$D(\rho) > 2k$	ρ_2
3	$D(\rho) \leq \overline{k}$	$D(\rho) > 2k$	ρ_3
4	$D(\rho) \leq \overline{k}$	$D(\rho) \in (k, 2k)$	ρ_4
5	$D(\rho) \leq \overline{k}$	$D(\rho) < k$	ρ_5

To avoid repeating the same statement, we start by stating that Lemma 7 implies that the corresponding first-order conditions for each of the five cases determine a unique solution for any parameter values and regions in question. The proof of the Lemma 7 appears at the end of the Appendix.

We continue by characterizing the strategy of the lowest type, $\rho(\underline{k})$. In particular, let

$$\rho^* = \arg\max_{\rho} \rho \int_{\underline{k}}^{\min\{D(\rho),\overline{k}\}} (D(\rho) - k) f(k) dk. \tag{6.1}$$

We show the following result:

Lemma 3. *In any monotone equilibrium, it holds that $\rho(\underline{k}) = \rho^*$ where ρ^* is uniquely defined by (6.1).*

Proof. First, we show uniqueness of ρ^*. Taking the first-order condition of (6.1), we obtain

$$D'(\rho)\rho + D(\rho) = \frac{\int_{\underline{k}}^{\min\{D(\rho),\overline{k}\}} k f(k) dk}{F(\min\{D(\rho), \overline{k}\})}. \tag{6.2}$$

Case 1: $D(\rho) < \overline{k}$.

Then we have $D'(\rho)\rho + D(\rho) = \frac{\int_{\underline{k}}^{D(\rho)} kf(k)dk}{F(D(\rho))}$. When D is concave the left-hand side is strictly decreasing with a derivative less than D'. It is then sufficient to prove that the right hand side has a derivative larger than D'. Let $x = D(\rho)$ and $h(x) = \frac{\int_{\underline{k}}^{x} kf(k)dk}{F(x)}$. Then the derivative of the right hand side is equal to $h'(x)D'(\rho)$. Therefore, if $h'(x) < 1$ for all x, then this derivative is between 0 and $D'(\rho)$, which is sufficient for our purposes.

First, $h'(x) = \frac{f(x)}{F(x)}(x - h(x))$, and then by the l'Hospital rule and $0 < f(\underline{k}) < \infty$, we obtain $h'(\underline{k}) = \frac{f(\underline{k})}{f(\underline{k})}(1 - h'(\underline{k}))$, so $h'(\underline{k}) = 1/2$. On the other hand, $h'(x) = 1$ implies $h''(x) = \left(\frac{f(x)}{F(x)}\right)'(x - h(x))$. It follows from F being concave that $\left(\frac{f(x)}{F(x)}\right)' < 0$, and thus $h'(x) = 1$ implies $h''(x) < 0$, which then implies together with $h'(\underline{k}) = 1/2$ that $h'(x) < 1$ for all $x > \underline{k}$. The above argument shows that there is at most a single solution of (6.1) if $D(\rho) < \overline{k}$ holds.

Case 2: $D(\rho) \geq \overline{k}$.

In this case, the right hand side is constant in ρ, so there is at most a single solution of (6.2) such that $D(\rho) > \overline{k}$ holds.

Combining the two cases immediately implies that there is a single solution, which has to be in one of those two regions.

Finally, the necessary condition that $\rho(\underline{k}) = \rho^*$ must hold follows from the same argument as in the proof of Proposition 3 in the main text. Q.E.D.

6.1.1 The case where $D(\rho(\overline{k})) < \overline{k}$

First, let us assume that $D(\rho^*) > 2\underline{k}$ and we will provide conditions under which $D(\rho(\overline{k})) < \overline{k}$. The profit function then becomes $\overline{\pi}(\rho, k) = F(D(\rho) - k)\rho k + \int_{D(\rho)-k}^{D(\rho)} f(x)(D(\rho) - x)dx\rho$ for values of k such that $D(\rho) > 2k$. We will use the following notation at times below: $\widehat{\gamma} = D(\rho) - k$, and $\eta = D(\rho)$. The relevant expected value is now

$$\tau = \frac{\int_{\widehat{\gamma}}^{\eta} f(x)xdx}{F(\eta) - F(\widehat{\gamma})}.$$

For future reference,

$$\frac{\partial \tau}{\partial \rho} = D'(\rho)\frac{f(\widehat{\gamma})(\tau - \widehat{\gamma}) + f(\eta)(\eta - \tau)}{F(\eta) - F(\widehat{\gamma})}, \tag{6.3}$$

and

$$\frac{\partial \tau}{\partial k} = -\frac{f(\widehat{\gamma})(\tau - \widehat{\gamma})}{F(\eta) - F(\widehat{\gamma})}. \tag{6.4}$$

The first derivative of π with respect to ρ can be written as $\frac{\partial \pi}{\partial \rho} = F(\widehat{\gamma})k + (F(\eta) - F(\widehat{\gamma}))(D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma}))$. Therefore, we obtain that

$$F(\gamma)k + (F(\eta) - F(\gamma))[D(\rho) + D'(\rho)\rho - \tau(\gamma)] = 0 \tag{6.5}$$

with the definition of $\gamma = D(\rho(k)) - k$.

Next, we need to show that for all $\rho < \rho(k)$ it holds that $\frac{\partial \pi}{\partial \rho} > 0$, and for all $\rho > \rho(k)$ it holds that $\frac{\partial \pi}{\partial \rho} < 0$. By the first order condition, equality holds at $\rho = \rho(k)$. These inequalities are unchanged if one divides by $F(\eta) - F(\widehat{\gamma})$ to show that $\frac{F(\widehat{\gamma})}{F(\eta) - F(\widehat{\gamma})}k + D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma})$ is decreasing in ρ. By concavity of the revenue function, $D(\rho) + D'(\rho)\rho$ is decreasing in ρ. Calculating shows

$$\frac{\partial}{\partial \rho}\left(\frac{F(\widehat{\gamma})}{F(\eta) - F(\widehat{\gamma})}k - \tau\right) =$$

$$= D'(\rho)\left(k\frac{f(\widehat{\gamma})(F(\eta) - F(\widehat{\gamma})) - f(\eta)F(\widehat{\gamma}) + F(\widehat{\gamma})f(\widehat{\gamma})}{(F(\eta) - F(\widehat{\gamma}))^2} - \frac{f(\widehat{\gamma})(\tau - \widehat{\gamma}) + f(\eta)(\eta - \tau)}{F(\eta) - F(\widehat{\gamma})}\right) =$$

$$= \frac{D'(\rho)}{F(\eta) - F(\widehat{\gamma})}[\frac{k\,(f(\widehat{\gamma})F(\eta) - f(\eta)F(\widehat{\gamma}))}{F(\eta) - F(\widehat{\gamma})} - (f(\widehat{\gamma})(\tau - \widehat{\gamma}) + f(\eta)(\eta - \tau))].$$

Therefore, to show that $\frac{F(\widehat{\gamma})}{F(\eta) - F(\widehat{\gamma})}k + D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma})$ is decreasing in ρ, it is sufficient to show that

$$\frac{k\,(f(\widehat{\gamma})F(\eta) - f(\eta)F(\widehat{\gamma}))}{F(\eta) - F(\widehat{\gamma})} \geq f(\widehat{\gamma})(\tau - \widehat{\gamma}) + f(\eta)(\eta - \tau),$$

which holds by $f(\widehat{\gamma}) \geq f(\eta)$ (by concavity of F), and by $\eta - \widehat{\gamma} = k$.[15]

We show that under our assumptions the solution of (6.5) is strictly decreasing. Recall that $\overline{\pi}(k, \rho) = F(D(\rho) - k)\rho k + \int_{D(\rho)-k}^{D(\rho)} f(x)(D(\rho) - x)dx\rho$, and that we have shown above that $\partial^2\overline{\pi}/\partial\rho^2 \mid_{\rho=\rho(k)} < 0$. Also, by the first order condition (6.5), $\partial\overline{\pi}/\partial\rho \mid_{\rho=\rho(k)} = 0$ for all $k \leq k^*$. The implicit function theorem applied to $\partial\overline{\pi}/\partial\rho \mid_{\rho=\rho(k)} = 0$ implies $\rho'(k) = -\frac{\partial^2\overline{\pi}/\partial\rho\partial k|_{\rho=\rho(k)}}{\partial^2\overline{\pi}/\partial\rho^2|_{\rho=\rho(k)}}$, and

[15]The display would hold as an equality when $f(\eta) = f(\widehat{\gamma})$, and decreasing $f(\eta)$ below $f(\widehat{\gamma})$ strengthens this direction.

thus for $\rho' < 0$ it is sufficient to show that $\partial^2\overline{\pi}/\partial\rho\partial k \mid_{\rho=\rho(k)} < 0$. Let $\varpi_0 = \frac{\partial\overline{\pi}/\partial\rho}{F(\eta)-F(\gamma)} = \frac{F(\gamma)}{F(\eta)-F(\gamma))}k + D(\rho) + D'(\rho)\rho - \tau$. By $\partial\overline{\pi}/\partial\rho \mid_{\rho=\rho(k)} = 0$, $\varpi_0 = 0$ when $\rho = \rho(k)$. Therefore, for $\partial^2\overline{\pi}/\partial\rho\partial k \mid_{\rho=\rho(k)} < 0$ it is sufficient to prove that $\varpi = \partial\varpi_0/\partial k \mid_{\rho=\rho(k)} \leq 0$, which becomes

$$\varpi = \frac{\partial}{\partial k}\left(\frac{F(\gamma)}{F(\eta)-F(\gamma)}k - \tau\right) \mid_{\rho=\rho(k)} \leq 0.$$

Rewriting this derivative, using also that $\frac{\partial\gamma}{\partial k} = -1$, we obtain the sufficient condition $\varpi = \frac{F(\gamma)}{F(\eta)-F(\gamma)} - (\frac{kf(\gamma)F(\eta)}{(F(\eta)-F(\gamma))^2} - \partial\tau/\partial k \mid_{\rho=\rho(k)}) \leq 0$, which can be rewritten (using (6.4)) as $\frac{kf(\gamma)F(\eta)}{F(\eta)-F(\gamma)} \geq F(\gamma) + f(\gamma)(\tau-\gamma)$. This can be rewritten as $kf(\gamma)F(\eta) \geq F(\gamma)(F(\eta)-F(\gamma)) + (F(\eta)-F(\gamma))f(\gamma)(\tau-\gamma)$, which holds if

$$kf(\gamma) \geq F(\eta) - F(\gamma),$$

and if

$$kf(\gamma) \geq f(\gamma)(\tau-\gamma).$$

By concavity of F, we have that $\frac{f(\gamma)(\eta-\gamma)}{F(\eta)-F(\gamma)} \geq 1$, which implies that $\frac{kf(\gamma)}{F(\eta)-F(\gamma)} \geq \frac{k}{\eta-\gamma}$. So, it is sufficient to prove that $\frac{k}{\eta-\gamma} \geq 1$, which holds due to $\gamma = D(\rho(k)) - k = \eta - k$. Second, $kf(\gamma) \geq f(\gamma)(\tau-\gamma)$ is equivalent to $k \geq \tau - \gamma$, which holds by $k + \gamma = D(\rho(k)) = \eta \geq \tau$.

This completes the analysis, and shows that global optimality conditions hold, and that the first order condition (6.5) determines a monotone price function ρ in the region where $D(\rho) > 2k$, and $D(\rho) \leq \overline{k}$.

Let k^* be defined as $\min_k\{D(\rho(k)) = 2k\}$. This leads us to the following result:

Lemma 4. *Assume that the type distribution F is concave, and the demand function is concave. If $D(p^*) > 2\underline{k}$, and $2k^* < \overline{k}$, then a symmetric, and monotone equilibrium consists of ρ_3 satisfying (6.5) for all $k \leq k^*$. Moreover, ρ_3 defined as in (6.5) is strictly decreasing, and satisfies the single crossing condition, so no profitable deviation exists for any type $k < k^*$ on the price interval $[\rho^*, \rho_3(k^*)]$.*

Lemma 7 implies that there is a unique value of k^*, so the upper bound of the type region where (6.5) is necessary—and sufficient as highlighted in Lemma

4— is well defined. For all $k > k^*$ the pricing problem changes, which implies a new first-order condition. We keep the assumption that $D(\rho(k)) < \overline{k}$ for now. The profit function can be written as

$$F(\widehat{k})\rho(\widehat{k})k + \int_{\widehat{k}}^{D(\rho(\widehat{k}))} f(x)D(\rho(\widehat{k})) - x)dx\rho(\widehat{k}).$$

The first order condition then becomes

$$\rho'(k)[F(k)k+(F(D(\rho))-F(k))\,(D(\rho) + D'(\rho)\rho - \tau(k))] = f(k)(D(\rho(k))-2k)\rho(k) \tag{6.6}$$

where $\tau(k) = \frac{\int_k^{D(\rho)} f(x)xdx}{F(D(\rho))-F(k)}$ is the expected value of the relevant types that type k loses the price war against.

By a similar argument as before, the single crossing condition becomes

$$F(k)(D(\rho) - k) + (F(D(\rho)) - F(k))[D'(\rho)\rho + D(\rho) - \tau(k)] \geq 0. \tag{6.7}$$

By construction,

$$\tau'(k) = \frac{f(k)(\tau - k)}{F(D(\rho)) - F(k)} + D'(\rho)\rho'(k)\frac{f(D(\rho))(D(\rho) - \tau)}{F(D(\rho)) - F(k)} \tag{6.8}$$

and $k < \tau < D(\rho)$.

Let $\delta = F(k)(D(\rho)-k)+(F(D(\rho))-F(k))[D'(\rho)\rho+D(\rho)-\tau(k)]$ be a short notation for the left hand side of (6.7). We need to show $\delta \geq 0$ for all k. Since $\delta(\underline{k}) = 0$ by the definition of $\rho(\underline{k}) = \widetilde{p}^*$, it is sufficient to show that for any k, $\delta(k) = 0 \Longrightarrow \delta'(k) \geq 0$. Taking the derivative, and letting the revenue function $R(p) = D(p)p$, we obtain

$$\delta' = f(k)(D(\rho) - k - [D'(\rho)\rho + D(\rho) - \tau(k)]) + F(k)(D'(\rho)\rho' - 1)$$

$$+f(D(\rho))D'(\rho)\rho'[D'(\rho)\rho + D(\rho) - \tau(k)] + (F(D(\rho)) - F(k))(R''(\rho)\rho' - \tau') =$$

$$f(k)(D(\rho) - k - [D'(\rho)\rho + D(\rho) - \tau(k)]) + F(k)(D'(\rho)\rho' - 1)$$

$$+f(k)(D(\rho))D'(\rho)\rho'[D'(\rho)\rho + D(\rho) - \tau(k)] + (F(D(\rho)) - F)R''(\rho)\rho'$$

$$-f(k)(\tau-k)-D'(\rho)\rho'(k)f(D(\rho))(D(\rho)-\tau).$$

using that $(F(D(\rho))-F(k))\tau' = f(k)(\tau-k)+D'(\rho)\rho'(k)f(D(\rho))(D(\rho)-\tau)$ from (6.8). Substituting in $\delta = 0$ now yields $-[D'(\rho)\rho+D(\rho)-\tau(k)] = \frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)}$. Then it can be further written when $\delta = 0$ as

$$\delta' = f(k)(D(\rho)-k+\frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)})-F(k)-f(\tau-k)$$

$$+(F(D(\rho))-F(k))R''(\rho)\rho'-D'(\rho)\rho'(k)f(D(\rho))(D(\rho)-\tau)$$

$$+F(k)D'(\rho)\rho'-f(D(\rho))D'(\rho)\rho'\frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)}$$

$$= f(k)(D(\rho_3)-\tau)+F(k)(f(k)\frac{D(\rho)-k}{F(D(\rho))-F(k)}-1)$$

$$+(F(D(\rho))-F)R''(\rho)\rho'-D'(\rho)\rho'(k)f(D(\rho))(D(\rho)-\tau)$$

$$+F(k)D'(\rho)\rho'-f(D(\rho))D'(\rho)\rho'\frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)}.$$

Next, we provide a condition under which the second line of the above expression is positive:

$$(F(D(\rho))-F)R''(\rho)\rho'-D'(\rho)\rho'(k)f(D(\rho))(D(\rho)-\tau)\geq 0,$$

or, using $\rho'(k) < 0$, that $(F(D(\rho))-F)R''(\rho) \leq D'(\rho)f(D(\rho))(D(\rho)-\tau)$. Dividing through by $D'(\rho)$, and using $R'' = 2D' + D''(\rho)\rho$ yields the equivalent condition, $2+\frac{D''(\rho(k))\rho(k)}{D'(\rho(k))} \geq \frac{f(D(\rho))(D(\rho)-\tau)}{F(D(\rho))-F(k)}$. By concavity of F, $\frac{f(D(\rho))(D(\rho)-\tau)}{F(D(\rho))-F(k)} \leq \frac{f(D(\rho))(D(\rho)-\tau)}{f(D(\rho))(D(\rho)-k)} = \frac{D(\rho)-\tau}{D(\rho)-k} \leq 1$. Therefore, it is sufficient to show $2+\frac{D''(\rho(k))\rho(k)}{D'(\rho(k))} \geq 1$, or that

$$D'(\rho)+D''(\rho)\rho \leq 0. \tag{6.9}$$

This condition is stronger than concavity of the revenue function but weaker than concavity of demand.

Also, the expression in the third line of the above expression is positive because

$$F(k)D'(\rho)\rho'-f(D(\rho))D'(\rho)\rho'\frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)} =$$

$$= F(k)D'(\rho)\rho'(k)(1 - f(D(\rho))\frac{D(\rho) - k}{F(D(\rho)) - F(k)}) \geq$$

$$F(k)D'(\rho)\rho'(k)(1 - \frac{F(D(\rho)) - F(k)}{F(D(\rho)) - F(k)}) = 0,$$

where the inequality is by concavity of F, which implies that $f(D(\rho))\,(D(\rho) - k) \leq F(D(\rho)) - F(k)$.

Given that $(F(D(\rho)) - F)R''(\rho)\rho' - D'(\rho)\rho'(k)f(D(\rho))(D(\rho) - \tau) \geq 0$ under (6.9), and $F(k)D'(\rho)\rho' - f(D(\rho))D'(\rho)\rho'\frac{F(k)(D(\rho)-k)}{F(D(\rho))-F(k)} \geq 0$ when F is concave, we obtain that

$$\delta' \geq f(k)(D(\rho) - \tau) + F(k)(f(k)\frac{D(\rho) - k}{F(D(\rho)) - F(k)} - 1)$$

when $\delta = 0$. Using that $D(\rho) > \tau$ we have that $\delta' > F(k)(f(k)\frac{D(\rho)-k}{F(D(\rho))-F(k)} - 1)$. Concavity of F implies $f(k)\frac{D(\rho)-k}{F(D(\rho))-F(k)} - 1 \geq 0$, which now establishes that $\delta' \geq 0$ when $\delta = 0$, and thus the single crossing condition (6.7) holds.

Given the above analysis, we obtain the following result:

Lemma 5. *Assume that F is concave, and that the demand function satisfies (6.9). If $D(\rho(k^*)) = 2k^* < \overline{k}$, then the pricing function ρ_4 that solves the differential equation (6.6) with initial condition $D(\rho(k^*)) = 2k^*$ is strictly decreasing, and satisfies the single-crossing condition for all k such that $D(\rho_4(k)) \geq k$. Any monotone and symmetric equilibrium is equal to ρ_4 for all $k > k^*$ such that $D(\rho_4(k)) \geq k$.*

Next, we characterize the pricing function for the region where $D(\rho(k)) < k$. Let $k^{**} \in (k^*, \overline{k})$ be the type such that $D(\rho_4(k^{**})) = k^{**}$. Define $g(p) = pD(p)F(D(p))$, and let $\rho_5(k)$ be such that

$$\rho_5(k)D(\rho_5(k))F(k) = \rho_5(k^{**})D(\rho_5(k^{**}))F(k^{**}) = D^{-1}(k^{**})k^{**}F(k^{**}). \quad (6.10)$$

for all $k > k^{**}$. Note, that if the revenue function is strictly concave, then $\rho_5(k)$ is uniquely determined from this formula because in the relevant range of prices below the monopoly price, it holds that $D(p)p$ is strictly increasing in p.

Let $\rho(k) = \rho_5(k)$ for all $k > k^{**}$ where function ρ_5 is defined by (6.10). Then the profit of type k when setting price $\rho(\widehat{k})$ is $\rho_5(\widehat{k})D(\rho_5(\widehat{k}))F(\widehat{k})$ because profits are zero upon setting a price higher than the opponent when $p < \rho_5(k^{**})$.

Therefore, it is optimal for a firm with type k to set price $\rho_5(k)$ as long as (6.10) determines a price function $\rho_5(k)$ such that $D(\rho_5(k)) \leq k$. We show that if $g(p) = pD(p)F(D(p))$ is concave in the price range where $p \leq \widetilde{p}^* = \rho(\underline{k})$, then $D(\rho_5(k)) \leq k$ indeed holds, and thus a monotone equilibrium is found.

Lemma 6. *Suppose that F, D are concave. Then (6.10) determines a solution such that $D(\rho_5(k)) \leq k$, and thus ρ_5 determines equilibrium pricing for all types $k > k^{**}$.*

Proof. Step 1. Suppose that $g'(\rho(k^{**})) \geq 0$, and g is concave in the relevant range $p \leq p^* = \rho(\underline{k})$. We show that $D(\rho_5(k)) \leq k$ for all $k \geq k^{**}$.

By concavity of g, $g'(\rho(k^{**})) \geq 0$ implies that for all $p \leq \rho(k^{**})$ it holds that $g'(p) \geq 0$, and thus for all $k > k^{**}$,

$$\rho_5(k)D(\rho_5(k))F(D(\rho_5(k))) \leq \rho(k^{**})D(\rho(k^{**}))F(D(\rho(k^{**}))).$$

Suppose that $D(\rho_5(k)) > k$. Then $\rho_5(k)D(\rho_5(k))F(D(\rho_5(k))) > \rho_5(k)D(\rho_5(k))F(k) = \rho(k^{**})D(\rho(k^{**}))F(k^{**}) = \rho(k^{**})D(\rho(k^{**}))F(D(\rho(k^{**})))$, so

$$\rho_5(k)D(\rho_5(k))F(D(\rho_5(k))) > \rho(k^{**})D(\rho(k^{**}))F(D(\rho(k^{**}))).$$

Since the two displays contradict each other, the step is complete.

Step 2. We show now that the way k^{**} is constructed implies $g'(\rho(k^{**})) \geq 0$.

Let us consider the profit function $\pi(\widehat{k}, k^{**})$ for $\widehat{k} < k^{**}$. This profit function is then according to $F(\widehat{k})\rho(\widehat{k})k + \int_{\widehat{k}}^{D(\rho(\widehat{k}))} f(x)D(\rho(\widehat{k})) - x)dx\rho(\widehat{k})$ as defined before (6.6). By the first order condition, it holds that

$$\lim_{\widehat{k} \nearrow k^{**}} \frac{\pi(k^{**}, k^{**}) - \pi(\widehat{k}, k^{**})}{k^{**} - \widehat{k}} = 0.$$

On the other hand, by construction, $D(\rho(k)) > k$ for all $k < k^{**}$. Therefore, $\pi(\widehat{k}, k^{**})$ is lower than in the hypothetical case where $D(\widehat{\rho}(k)) = k$ for all $k < k^{**}$ because this second strategic situation represents less aggressive play by the opponent. Let $u(\widehat{k}, k^{**})$ denote the profit of the firm with type k^{**} if it sets a price of $\widehat{\rho}(\widehat{k}) = D^{-1}(\widehat{k})$. Formally, the profit comparison can be written as

$$u(\widehat{k}, k^{**}) \geq \pi(\widehat{k}, k^{**}).$$

It follows by construction, that

$$u(\widehat{k}, k^{**}) = D^{-1}(\widehat{k})\widehat{k}F(\widehat{k}) = \widehat{p}D(\widehat{p})F(D(\widehat{p})) = g(\widehat{\rho}(\widehat{k}))$$

with $\widehat{\rho}(\widehat{k}) > \widehat{\rho}(k^{**}) = \rho(k^{**})$. Also, by construction,

$$\pi(k^{**}, k^{**}) = g(\rho(k^{**})) = g(\widehat{\rho}(k^{**})).$$

Putting together the last four displays implies that $\lim_{\widehat{k} \nearrow k^{**}} \frac{g(\widehat{\rho}(k^{**})) - g(\widehat{\rho}(\widehat{k}))}{k^{**} - \widehat{k}} = g'(\widehat{\rho}(k^{**}))\widehat{\rho}'(k^{**}) \leq 0$. This result, together with $\widehat{\rho}' < 0$, implies that $g'(\widehat{\rho}(k^{**})) = g'(\rho(k^{**})) \geq 0$, which concludes the proof of Step 2.

Step 3. We show that when F, D are both concave, function $g = RF(D)$ is concave in the relevant range where $R'(p) > 0$.

Note, that

$$g''(p) = R''(p)F(D(p)) + 2R'(p)f(D(p))D'(p) + R(p)\left(f'(D(p))\left(D'(p)\right)^2 + f(D(p))D''(p)\right).$$

When F, D are both concave (and thus R is concave as well), it holds that $g'' \leq 0$. Q.E.D.

We have constructively defined the following pricing function together with the unique cutoffs: $\rho = \rho_3$ for $k \leq k^*$; $\rho = \rho_4$ for $k \in (k^*, k^{**}]$; and $\rho = \rho_5$ for $k > k^{**}$. Let us denote this piecewise pricing function by ρ^I. If the above defined k^{**} satisfies $k^{**} < \overline{k}$, then it indeed holds that $D(\rho^I(\overline{k})) < \overline{k}$. Moreover, all the incentive and monotonicity conditions hold, so ρ^I constitutes a monotone, symmetric equilibrium.

However, given that firms with high capacities do not use their entire capacity level in equilibrium, types that are close to each other are indifferent between nearby prices. Therefore, a continuum of non-monotone equilibria also exist.

We summarize this result in the following Proposition:

Proposition 6. *If $D(\rho^*) > 2\underline{k}$, and $k^{**} < \overline{k}$, then there exists a symmetric monotone equilibrium, and a continuum of non-monotone equilibria.*

To complete the analysis for the case where $D(\rho(\overline{k})) < \overline{k}$, we also need to consider the case where $D(\rho^*) \leq 2\underline{k}$. It is clear that $D(\rho^*) > \underline{k}$, and thus we only need to consider the case where $D(\rho^*) \in (\underline{k}, 2\underline{k})$. However, the exact same analysis applies as above except that the region where the pricing function takes the form of ρ_3 is now empty (or $k^* = \underline{k}$). Therefore, we obtain the following result.

Corollary 1. *If $k^{**} < \overline{k}$, then there exists a symmetric and monotone equilibrium.*

6.1.2 The case where $D(\rho(\overline{k})) > \overline{k}$

In this case, we will provide an equilibrium where $D(\rho) > k$ for k. First, let us assume that $D(\rho^*) \geq \overline{k}$. Then the analysis from the main text applies, and we have an equilibrium that involves only function ρ_1 if $D(\rho^*) \leq 2\underline{k}$ or it involves both ρ_1 and ρ_2 if $D(\rho^*) > 2\underline{k}$ as Proposition 4 states.

Next, let us assume $D(\rho^*) < \overline{k}$. We focus on the case where $D(\rho^*) > 2\underline{k}$ because the case where $D(\rho^*) \leq 2\underline{k}$ can be handled in a straightforward manner as the pricing function has one less region in that case.

Let us recall the constructions for ρ_3, ρ_4 and k^* from above. In particular, ρ_4 is the pricing function characterized by the first order conditions when $D(\rho) < \overline{k}$, $D(\rho) \in (k, 2k)$, and ρ_3 is the pricing function characterized by the first order conditions when $D(\rho) < \overline{k}$, $D(\rho) > 2k$. Also, k^* is defined implicitly by $D(\rho_3(k^*)) = 2k^*$. Such a $k^* \in (\underline{k}, \overline{k})$ must exist when $D(\rho^*) = D(\rho(\underline{k})) > 2\underline{k}$ because we have established it above that $D(\rho_3(\overline{k})) < 2\overline{k}$.

Case 1: Suppose that $D(\rho_3(k^*)) = 2k^* \leq \overline{k}$.

Then there exists $k^+ > k^*$ and $D(\rho_4(k^+)) = \overline{k}$ because otherwise $D(\rho(\overline{k})) > \overline{k}$ could not hold. Then for all $k \in (k^*, k^+]$ the candidate equilibrium takes the form of $\rho = \rho_4$, while for $k > k^+$ the analysis from the main text establishing Proposition 3 applies. In particular, for $k > k^+$ it holds that $D(\rho) \in (k, 2k)$, and $D(\rho) > \overline{k}$ and thus pricing function ρ_1 from the main text applies with the initial condition $D(\rho_1(k^+)) = \overline{k}$. Given that in all three regions (up to k^*, between k^* and k^+, and above k^+) the prior analysis has already established monotonicity of the candidate equilibrium pricing functions ρ_3, ρ_4, ρ_1, and the single crossing conditions, so a symmetric, monotone equilibrium exists in this case. In particular, the equilibrium pricing function is ρ_3, ρ_4 and ρ_1 in the three regions with the appropriate initial conditions that ensure continuity of ρ.

Case 2: Suppose that $D(\rho_3(k)) > 2k$ for all k such that $D(\rho) \leq \overline{k}$.

Then there exists a type $k^+ < k^*$ such that $D(\rho_3(k^+)) = \overline{k} > 2k^+$, and the equilibrium strategy takes the form of $\rho = \rho_3$ for $k \in [\underline{k}, k^+]$. For $k > k^+$ such that $D(\rho(k)) > 2k$ still holds, we characterize the pricing function ρ_2

used in Proposition 4 for the region of types where $D(\rho) > 2k$, and $D(\rho) > \overline{k}$. We conduct this analysis by simplifying the analysis of the Appendix that characterizes function ρ_3, which pertains to the case where $D(\rho) > 2k$, and $D(\rho) < \overline{k}$.

In this case, the profit function becomes

$\overline{\pi}(\rho, k) = F(D(\rho) - k)\rho k + \int_{D(\rho)-k}^{\overline{k}} f(x)(D(\rho) - x)dx\rho$. Recall the following notation form above: $\widehat{\gamma} = D(\rho) - k$. The relevant expected value is now

$$\tau = \frac{\int_{\widehat{\gamma}}^{\overline{k}} f(x)xdx}{1 - F(\widehat{\gamma})}.$$

For future reference,

$$\frac{\partial \tau}{\partial \rho} = D'(\rho)\frac{f(\widehat{\gamma})(\tau - \widehat{\gamma})}{1 - F(\widehat{\gamma})},$$

and

$$\frac{\partial \tau}{\partial k} = -\frac{f(\widehat{\gamma})(\tau - \widehat{\gamma})}{1 - F(\widehat{\gamma})}.$$

The first derivative of π with respect to ρ can be written as $\frac{\partial \overline{\pi}}{\partial \rho} = F(\widehat{\gamma})k + (F(\eta) - F(\widehat{\gamma}))(D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma}))$. Therefore, the first order condition is

$$F(\gamma)k + (1 - F(\gamma))[D(\rho) + D'(\rho)\rho - \tau(\gamma)] = 0 \tag{6.11}$$

with the definition of $\gamma = D(\rho(k)) - k$. Note, that the first order condition (6.11) is the same as (6.5) with the $\eta = \overline{k}$ substitution as η' does not appear in (6.5).

To show that the pricing function ρ_4 that is the solution of (6.11) is strictly decreasing, one can follow the same steps as before Lemma 4 to see that now sufficient conditions are as follows:

$$kf(\gamma) \geq 1 - F(\gamma),$$

and

$$kf(\gamma) \geq f(\gamma)(\tau - \gamma).$$

Then using the same steps again, we need to show that $\gamma + k \geq \overline{k}$, which holds via $\gamma + k = D(\rho(k)) > \overline{k}$ because $k > k^+$. Second, $kf(\gamma) \geq f(\gamma)(\tau - \gamma)$ is equivalent to $k \geq \tau - \gamma$, which holds by $k + \gamma = D(\rho(k)) \geq \overline{k} \geq \tau$.

Next, we need to show that for all $\rho < \rho(k)$ it holds that $\frac{\partial \pi}{\partial \rho} > 0$, and for all $\rho > \rho(k)$ it holds that $\frac{\partial \pi}{\partial \rho} < 0$. By the first order condition (6.11), equality holds at $\rho = \rho(k)$. These inequalities are unchanged if one divides by $1 - F(\widehat{\gamma})$ to show that $\frac{F(\widehat{\gamma})}{1-F(\widehat{\gamma})}k + D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma})$ is decreasing in ρ. By concavity of the revenue function, $D(\rho) + D'(\rho)\rho$ is decreasing in ρ. Calculating shows

$$\frac{\partial}{\partial \rho}\left(\frac{F(\widehat{\gamma})}{1 - F(\widehat{\gamma})}k - \tau\right) =$$

$$= D'(\rho)\left(k\frac{f(\widehat{\gamma})(1 - F(\widehat{\gamma})) + F(\widehat{\gamma})f(\widehat{\gamma})}{(1 - F(\widehat{\gamma}))^2} - \frac{f(\widehat{\gamma})(\tau - \widehat{\gamma})}{1 - F(\widehat{\gamma})}\right) =$$

$$= \frac{D'(\rho)}{1 - F(\widehat{\gamma})}[\frac{kf(\widehat{\gamma})}{1 - F(\widehat{\gamma})} - f(\widehat{\gamma})(\tau - \widehat{\gamma})].$$

Therefore, to show that $\frac{F(\widehat{\gamma})}{1-F(\widehat{\gamma})}k + D(\rho) + D'(\rho)\rho - \tau(\widehat{\gamma})$ is decreasing in ρ, it is sufficient to show that

$$\frac{k}{1 - F(\widehat{\gamma})} \geq \tau - \widehat{\gamma},$$

which holds by $\widehat{\gamma} + k \geq \overline{k} \geq \tau$.

The above function ρ_2 characterizes a type k^{++} such that $D(\rho_2(k^{++})) = 2k^{++} > \overline{k}$. For all $k > k^{++}$ again the analysis for Proposition 3 applies which derived pricing function ρ_1, so the incentive conditions hold for the constructed candidate equilibrium pricing function. In summary, the pricing function is then ρ_3, ρ_2, ρ_1 for the regions $[\underline{k}, k^+), [k^+, k^{++}), [k^{++}, \overline{k}]$ respectively with all the necessary monotonicity and incentive conditions in place.

Finally, when $D(\rho^*) < 2\underline{k}$, then a simplified analysis applies where only pricing functions ρ_4, ρ_1 appear in the equilibrium pricing function. Q.E.D.

6.1.3 Existence and uniqueness of the solution of the initial value problems

To complete the proof of our results, we need to show that a unique solution exists for all the five different pricing functions $\rho_1, ..., \rho_5$ as determined by the first order conditions. Auction-like competition between the two firms only arises when $D(\rho) \in (k, 2k)$ for the following reason. First, if $D(\rho) > 2k$, then even the losing firm can produce at full capacity on the margin—when losing against a firm close to its own capacity level. Second, if $D(\rho) < k$, then capacities are not utilized fully even when winning. In both of these last two cases, competition is either completely absent or the optimal price does not depend on one's own type. In these cases, instead of a differential equation, the first order condition provides the value of the price ρ implicitly, and we can use the implicit function to obtain existence and uniqueness. This is the case for ρ_2, ρ_3, ρ_5 (see (4.1), (6.5) and (6.11)).

In the other two cases, the differential equation hits a singularity point at $\underline{k}$. In these cases, we cannot use the fundamental theorem of ordinary differential equations but standard techniques nonetheless allow one to obtain the existence of a solution. For details, see for example Kelevedjiev (2000) who constructs a solution (which is the highest solution in our case) explicitly. This is the case for ρ_1, ρ_4.

6.1.4 Existence and uniqueness for functions ρ_2, ρ_3 and ρ_5

For ρ_2, ρ_3,and ρ_5 using the Implicit Function Theorem directly makes the proof of existence and uniqueness fairly straightforward.

1. Existence and uniqueness for ρ_2

Expression (4.1) is

$$F(\gamma)k + (1 - F(\gamma))\left[D(\rho) + D'(\rho)\rho - \tau(\gamma)\right] = 0 = \Gamma(\rho, k)$$

Using the implicit function theorem one finds:

$$\rho' = -\frac{\Gamma_k}{\Gamma_\rho} = -\frac{F(\gamma) + f(\gamma)D'(\rho)\rho}{(1 - F(\gamma))[2D'(\rho) + D''(\rho)\rho] - f(\gamma)(D'(\rho))^2\rho}$$

where we used that $\gamma = D(\rho) - k$ and $(1 - F(\gamma)\tau(\gamma) = \int_\gamma^{\overline{k}} f(x)x dx$, and thus

$$\Gamma_k = F(\gamma) - f(\gamma)k + f(\gamma)\left[D(\rho) + D'(\rho)\rho\right] - \partial\left((1 - F(\gamma))\tau(\gamma)\right)/\partial k =$$

$$= F(\gamma) - f(\gamma)k + f(\gamma)\left[D(\rho) + D'(\rho)\rho\right] - f(\gamma)\gamma = F(\gamma) + f(\gamma)D'(\rho)\rho.$$

Under our assumption of concave revenues, the denominator is strictly negative. Therefore, fixing any initial condition, the fundamental theorem of ordinary differential equations implies that there is a unique solution.

2. Existence and uniqueness for ρ_3

Expression (6.5) is

$$F(\gamma)k + (F(\eta) - F(\gamma))[D(\rho) + D'(\rho)\rho - \tau(\gamma)] = 0 = \Gamma(\rho, k)$$

The implicit function theorem then yields:

$$\rho' = -\frac{\Gamma_k}{\Gamma_\rho} = -\frac{F(\gamma) + f(\gamma)D'(\rho)\rho}{(F(\eta) - F(\gamma))[2D'(\rho) + D''(\rho)\rho] + (D'(\rho))^2\rho(f(\eta) - f(\gamma))}$$

The denominator is always negative since $f(\eta) < f(\gamma)$ by concavity of F, and $F(\eta) > F(\gamma)$ because $\eta > \gamma$. Therefore, fixing any initial condition, the fundamental theorem of ordinary differential equations implies that there is a unique solution.

3. Existence and uniqueness for ρ_5

Equation (6.10) is

$$\rho_5(k)D(\rho_5(k))F(k) = \rho_5(k^{**})D(\rho_5(k^{**}))F(k^{**}) = D^{-1}(k^{**})k^{**}F(k^{**}) = C$$

where C is a constant. Using the implicit function theorem one finds:

$$\rho' = -\frac{f(k)\rho D(\rho)}{F(k)[\rho' D(\rho) + D(\rho)]}$$

When D is concave then the denominator is always defined and positive because for the relevant range of prices below the monopoly price, revenue is increasing in price. Hence the expression for ρ_5' is well defined (and negative). Therefore, fixing any initial condition, the fundamental theorem of ordinary differential equations implies that there is a unique solution.

6.1.5 Existence and uniqueness for functions ρ_1 and ρ_4

We have already argued for the existence of a solution above, so we only need to cover uniqueness of the solution here. For ρ_1, and $\rho_4$the proof utilizes stability conditions on the differential equation to show that uniqueness must hold despite the singularity point at $\underline{k}$.

4. Uniqueness for ρ_1

Let us start with the differential equation (3.5):

$$\rho' = \frac{(D(\rho) - 2k)f(k)\rho(k)}{F(k)k + (1 - F(k))[D'(\rho)\rho + D(\rho) - \tau(k))]}.$$

Note, that the numerator (which is negative in the baseline case) is decreasing in ρ, while the denominator (which is positive) is also decreasing in ρ by concavity of the revenue function. Therefore, the entire fraction is decreasing in ρ.

Now, fix any $k > \underline{k}$, and take two initial value conditions $\rho(k) = p_1$, and $\rho(k) = p_2$ with $p_2 > p_1$. Suppose that the initial value problem of (3.5), and these two initial conditions both lead to a solution where $\rho(\underline{k}) = p^*$, which would mean non-uniqueness of the equilibrium. We will show that this leads to a contradiction. Given the observation above,

$$\rho'_{(2)}(k) < \rho'_{(1)}(k) < 0$$

where $\rho_{(i)}$ corresponds to the solution where $\rho(k) = p_i$. This implies that $\rho_{(2)}$ is steeper than $\rho_{(1)}$ and thus for all $k' < k$ it holds that $\rho_{(2)}(k') - \rho_{(1)}(k') > p_2 - p_1$. But then it cannot occur that $\rho_{(2)}(\underline{k}) = \rho_{(1)}(\underline{k}) = p^*$. Q.E.D.

5. Uniqueness for ρ_4

Let us start with the differential equation (6.6):

$$\rho' = \frac{(D(\rho) - 2k)f(k)\rho(k)}{F(k)k + (F(D(\rho)) - F(k))[D'(\rho)\rho + D(\rho) - \tau(k))]}.$$

Note, that the numerator (which is negative in case 4) is decreasing in ρ, while the denominator (which is positive) is also decreasing in ρ by concavity of the revenue function. Therefore, the entire fraction is decreasing in ρ.

Given this, the rest of the argument is identical to the one we utilized for function ρ_1. Q.E.D.

6.1.6 General result for existence and uniqueness

Given the above, we have shown the following result:

Lemma 7. *For all five pricing functions $\rho_1, ..., \rho_5$ as defined by the initial value problems for ρ_1, ρ_4 and as defined by the implicit functions for ρ_2, ρ_3, and ρ_5 together with the relevant initial conditions, there exists a unique solution.*

This concludes the proof of Proposition 2.

7 Online Appendix 1: A case with two types where monotonicity fails

7.1 Setup

Assume $D = 1 - p$, and take the case where both firms have a low capacity of $k < 1/3$ that is common knowledge. In this case, both firms charge $\rho = 1 - 2k$ in equilibrium, and $D(\rho) = 2k$, which means that the firms utilize their entire capacity. However, if firms have low, and heterogeneous capacities with two possible types ($\underline{k} < \overline{k} < 1/3$), then it is easy to show that a pure strategy equilibrium with full capacity utilization does not exist because at least one of the two types would have a profitable local deviation. Moreover, when capacities are low ($\underline{k} < 1/3$), a low capacity firm utilizes its full capacity in equilibrium when the other firm has a low capacity as well. Therefore, low capacity firms do not compete against each other, which implies that a monotone equilibrium, if such an equilibrium exists, takes the following form:

1. A low capacity firm sets price $\overline{\rho}$ with probability one where $D(\overline{\rho}) \geq 2\underline{k}$. Note, that this implies that a low capacity firm utilizes its entire capacity when the opponent has a low capacity as well.

2. A high capacity firm randomizes on $[\underline{\rho}, \overline{\rho})$ according to an atomless distribution function G.

In our numerical example there are two types $\underline{k} = 0.2$, and $\overline{k} = 0.3$. Let β be the probability of the low capacity type $\underline{k}$. By point 1 above and the argument given in the baseline case when deriving p^*, the problem of the low capacity firm type is $\max_{p\in[0.5,0.6]} p(0.2\beta + (1 - p - 0.3)(1 - \beta))$. This implies that the optimal price for such a firm is $p = \frac{0.7-0.5\beta}{2(1-\beta)}$ if $\beta \in [0.6, 5/7]$, $p = 0.5$ if $\beta \leq 0.6$, and $p = 0.6$ if $\beta \geq 5/7$.

To construct our example, we focus on the case where $\overline{\rho} = 0.6$ (that is, where $\beta \geq 5/7$). For any $p < \overline{\rho}$, let $G(p)$ denote the probability that the high type chooses a price less than p. Then the sales of the high capacity firm equal to $S_H = 0.3(1 - G(p)(1 - \beta)) + (D(p) - 0.3)G(p)(1 - \beta)$, and the sales of the low capacity firm equal to $S_L = 0.2(1 - G(p)(1 - \beta)) + (D(p) - 0.3)G(p)(1 - \beta)$. Since the high type randomizes on $[\underline{\rho}, \overline{\rho})$, it holds that

$$pS_H(p) = \overline{\rho}S_H(\overline{\rho}) \tag{7.1}$$

for all $p \in [\underline{\rho}, \overline{\rho})$. The incentive condition for the low type to charge $\overline{\rho}$ is $pS_L(p) \leq \overline{\rho}S_L(\overline{\rho})$. Therefore, for all $p < \overline{\rho}$, it needs to hold that $\frac{pS_H(p)}{pS_L(p)} \geq \frac{\overline{\rho}S_H(\overline{\rho})}{\overline{\rho}S_L(\overline{\rho})}$, or

$$\frac{0.3(1 - G(p)(1 - \beta)) + (D(p) - 0.3)G(p)(1 - \beta)}{0.2(1 - G(p)(1 - \beta)) + (D(p) - 0.3)G(p)(1 - \beta)} > \frac{0.3\beta + 0.1(1 - \beta)}{0.2\beta + 0.1(1 - \beta)}.$$

This formula states that the increase in sales that a low capacity firm achieves by decreasing its price (from $\overline{\rho}$ to $p < \overline{\rho}$) is lower compared to the high capacity firm. This single crossing condition can be rewritten as $0.01(1 - G(p)(1 - \beta))(1 - \beta) > 0.1\beta(D(p) - 0.3)G(p)(1 - \beta)$ or $0.1(1 - G(p)(1 - \beta)) > \beta(D(p) - 0.3)G(p)$. Using $D = 1 - p$, and rearranging

$$G(p) < \frac{1}{1 + \beta(6 - 10p)}. \tag{7.2}$$

The indifference condition (7.1) implies that for all $p < \overline{\rho}$,

$$p[0.3(1 - G(p)(1 - \beta)) + (0.7 - p)G(p)(1 - \beta)] = 0.6[0.3\beta + 0.1(1 - \beta)],$$

or $\frac{0.6[0.3\beta+0.1(1-\beta)]}{p} = 0.7 - p + (p - 0.4)(1 - G(p)(1 - \beta))$, which can be solved to obtain

$$G(p) = \frac{1 - \frac{\frac{0.6[0.3\beta+0.1(1-\beta)]}{p} - (0.7-p)}{p-0.4}}{1 - \beta}. \tag{7.3}$$

For a monotone equilibrium to exist, we need to show that (7.3) implies (7.2).

For $\beta = 0.8$ the single crossing condition (7.2) becomes

$$\frac{1}{1+0.8(6-10p)} - \frac{1-\frac{\frac{0.6(0.3*0.8+0.1(1-0.8))}{p}-(0.7-p)}{p-0.4}}{1-0.8} \geq 0.$$

Plotting this condition yields the following graph:

This shows that the single crossing condition fails for p close to $\overline{\rho}$. There are two opposing effects that decide whether the single crossing condition holds. On one hand, a high capacity firm has more to gain in sales from winning because it sells more when winning ($\overline{k}$ vs $\underline{k}$) but the same quantity when losing ($D(p) - \overline{k}$). On the other hand, sales at the baseline price at (slightly below) $\overline{\rho}$ are also higher for the high capacity firm, $0.3\beta + 0.1(1-\beta)$ versus $0.2\beta + 0.1(1-\beta)$. It is easy to show that charging a price close to $\underline{\rho}$ is not profitable for the firm with a low capacity but charging a price close to (and below) $\overline{\rho}$ may be profitable for the low type depending on parameter values. When β is close to one $\overline{\rho} - \underline{\rho}$ is small, which implies that there is no profitable deviation near price $\overline{\rho}$ either; see the remark below.

Remark: If $\beta = 0.85$ is assumed instead of $\beta = 0.8$, then the single-crossing condition holds, and a monotone equilibrium of the form conjectured exists. The cutoff for a profitable local deviation at $p = \overline{\rho} = 0.6$ is at $\beta = 5/6$: when $\beta < 5/6$ setting a price slightly lower than $\overline{\rho}$ is a profitable (local) deviation for a firm with a low capacity. Given this, a monotone equilibrium exists when either the high or the low type is very likely but existence may fail for intermediate probabilities.

7.2 A non-monotone equilibrium when $\beta = 0.8$

So, take $\beta = 0.8$ and let us construct an equilibrium where the low type mixes on $[\widetilde{p}, 0.6]$, and the high type mixes on $[\underline{p}, 0.6]$ where $\underline{p} < \widetilde{p}$. For all $p \in [\widetilde{p}, 0.6]$,

$$p(0.2(1-(1-\beta)G_H) + (0.7-p)(1-\beta)G_H = U_L, \tag{7.4}$$

and

$$p(0.3(1-\beta G_L - (1-\beta)G_H) + (0.8-p)\beta G_L + (0.7-p)(1-\beta)G_H) = U_H.$$

Letting, $\Delta = U_H - U_L$ we have

$$\Delta = p(0.1(1-(1-\beta)G_H) + \beta G_L(0.5-p)). \tag{7.5}$$

We need Δ and U_L to be constant on $[\widetilde{p}, 0.6]$. Note, that (7.4) pins down G_H upon noticing that $U_L = 0.6(0.2*0.8+0.1*0.2) = 0.108$.

Note, that $\Delta = \Delta(0.6) = 0.6(0.1*0.8 - 0.8G_L(0.6)*0.1) = 0.048(1-G_L(0.6))$ where $G_L(0.6)$ is the probability that the low type chooses a price strictly less than 0.6. Then (7.5), and $\Delta = 0.048(1-G_L(0.6))$ pin down G_L (note that G_H was already pinned down from (7.4)) for a fixed value of $G_L(0.6)$.

Then (7.5) also implies

$$0.1\widetilde{p}(1-(1-\beta)G_H(\widetilde{p})) = 0.048(1-G_L(0.6)),$$

so $G_L(0.6)$ also pins down $\widetilde{p}$ because the function G_H is know from above.

So, the entire solution is pinned down by $\widetilde{p}$ (or equivalently by $G_L(0.6)$). To determine $\widetilde{p}$ we use a smooth pasting condition. By definition $G_L(\widetilde{p}) = 0$. To have a smooth change in the incentive condition for the high type we require that $g_L(\widetilde{p}) = 0$ holds as well. The rest of the calculation then derives the value of $\widetilde{p}$ and completes the calculations to establish our mixed strategy equilibrium with overlapping support, i.e. a non-monotone equilibrium. From (7.4), $0.2G_H = \frac{0.2-\frac{1.08}{p}}{p-0.5}$. Then from (7.5), and $\Delta = 0.1\widetilde{p}(1-(1-\beta)G_H(\widetilde{p})) = 0.1\widetilde{p}(1-\frac{0.2-\frac{1.08}{\widetilde{p}}}{\widetilde{p}-0.5})$, we have

$$0.1\widetilde{p}(1-\frac{0.2-\frac{1.08}{\widetilde{p}}}{\widetilde{p}-0.5}) = \Delta = p(0.1(1-\frac{0.2-\frac{1.08}{p}}{p-0.5}) + 0.8G_L(0.5-p)).$$

This implies

$$0.8G_L = \frac{\frac{\Delta}{p} - 0.1(1-\frac{0.2-\frac{1.08}{p}}{p-0.5})}{0.5-p}.$$

We are looking for a value $\widetilde{p}$ such that $G_L = G_L' = g_L = 0$. With simplifications this requires $\left(\frac{\Delta}{p} - 0.1(1 - \frac{0.2-\frac{1.08}{p}}{p-0.5})\right)' = 0$ or

$$\frac{0.1}{(\widetilde{p}-0.5)^2}(\frac{1.08}{\widetilde{p}^2}(\widetilde{p}-0.5) - 0.2 + \frac{1.08}{\widetilde{p}}) = \frac{\Delta}{\widetilde{p}^2} =$$

$$= \frac{0.1(1 - \frac{0.2-\frac{1.08}{\widetilde{p}}}{\widetilde{p}-0.5})}{\widetilde{p}},$$

where the second equality comes from $G_L = 0$. This can be solved numerically to obtain $\widetilde{p} = 0.5895$. Then plugging back we obtain $G_L(0.6) = 0.0023$. So, the low type chooses $p = 0.6$ with a probability of 99.77%, and chooses a price strictly below 0.6 with a probability of 0.23%. The lowest price ever chosen by the low capacity type is 0.5895. The high type achieves a utility of $0.108 + \Delta = 0.108 + 0.048 * 0.9977 = 0.1559$. The lowest price on the support for the high type is then $0.1559/0.3 = 0.5197$. We can also derive $G_H(\widetilde{p}) = 0.9382$. Note that the overall mass of high type strategies is $0.2 * 6.18\% = 1.24\%$, while the overall weight for low type strategies is $0.8 * 0.23\% = 0.18\%$ on price interval $[0.5895, 0.6)$, so the low type represents a non-negligible relative mass of around $0.18/1.42 = 13\%$ on the interval with overlapping strategies.

8 Online Appendix 2: Elastic demand and monotone equilibrium

When demand is inelastic Fabra and Llobet (2022) show that in a discriminatory auction there exists a decreasing price equilibrium independent of the type distribution F. We now argue, by means of a generalization of our Example 1 in Section 3.2, that introducing price sensitivity their result no longer necessarily holds, even for relatively inelastic demands.

Example 2. Let $D(p) = 1 - ap$ if $p \in [0, 1]$ and $D(p) = 0$ if $p > 1$, where $a \in (0, 1]$, and assume that firm 1 has capacity $k_1 = 0.75$, and firm 2 has capacity $k_2 = 0.5$. The discriminatory auction discussed in Fabra and Llobet (2022) is obtained by setting $a = 0$. The equilibrium can be characterized as follows.

Case 1: $a \in (1/4, 1]$.

For any $a \in (1/4, 1]$, both firms charge prices on interval $[\frac{1}{12a}, \frac{1}{4a}]$ with distribution functions $F_1 = \frac{12ap-1}{6ap(1+4ap)}$ and $F_2 = 3F_1/2$. Note, that the firm with the higher capacity places an atom of $1/3$ on price $p = 1/4a$. Writing the indifference condition of firm 2, it holds for all $p \in [\frac{1}{12a}, \frac{1}{4a})$ that

$$0.5p(1 - F_1(p)) + (1 - p - k_1)pF_1(p) = \pi^*$$

where π^* is the equilibrium profit of firm 2.

Let us now consider the incentives of a dummy type of firm 2 by introducing a zero probability type k not equal to 0.5. Then taking the equilibrium from above, the profit of dummy type k of firm 2 from charging price p is

$$p(1 - F_1(p))k + pF_1(p)(1 - p - k_1) = \pi^* + p(1 - F_1(p))(k - 0.5)$$

where

$$p(1 - F_1(p)) = \frac{1 - 6ap + 24a^2p^2}{6a(1 + 4ap)}$$

In our example, it is easy to show that $p(1 - F_1(p))$ is convex. This implies that $p(1 - F_1(p))(k - 0.5)$ is convex in p when $k > 0.5$. It takes its maximal value at both $p = \frac{1}{12a}$ and $p = \frac{1}{4a}$.

However, when $k < 0.5$, $p(1 - F_1(p))(k - 0.5)$ is concave, and the new type's profits are maximized at an interior price $p \in (\frac{1}{12a}, \frac{1}{4a})$. This contradicts monotonicity of incentives because a lower capacity does not lead to choosing the highest price $p = 1/4a$ but instead leads to choosing an intermediate price.

Case 2. For any $a \in (0, 1/4]$

Both firms charge prices on interval $[\frac{2}{3}(1 - 2a), 1]$ with distribution functions $F_1 = \frac{6p-4+8a}{3p(1+4ap)}$ and $F_2 = 3F_1/2$.

Note again that the firm with the higher capacity places an atom of $1/3$ on price $p = 1$. Writing the indifference condition of firm 2, it holds for all $p \in [\frac{2}{3}(1 - 2a), 1)$ that

$$0.5p(1 - F_1(p)) + (1 - p - k_1)pF_1(p) = \pi^*$$

where π^* is the equilibrium profit of firm 2.

Let us again consider the incentives of a dummy type of firm 2 by introducing a zero probability type k not equal to 0.5. Then taking the equilibrium from above, the profit of dummy type k of firm 2 from charging price p is

$$p(1 - F_1(p))k + pF_1(p)(1 - p - k_1) = \pi^* + p(1 - F_1(p))(k - 0.5)$$

where

$$p(1 - F_1(p)) = \frac{12ap^2 - 3p - 8a + 4}{3(1 + 4ap)}$$

It is again easy to show that $p(1 - F_1(p))$ is convex. This implies that $p(1 - F_1(p))(k - 0.5)$ is convex in p when $k > 0.5$, moreover, it is easy to check that it is maximized at $p = \frac{2}{3}(1 - 2a)$. In this case, the monotonicity of firm 2's incentive holds because a higher capacity yields to an optimal choice of a lower price.

However, when $k < 0.5$, $p(1 - F_1(p))(k - 0.5)$ is concave, and the new type's profits are maximized at an interior price $p \in (\frac{2}{3}(1 - 2a), 1)$ for $a \in (0.15, 0.25]$. This contradicts monotonicity of incentives because a lower capacity does not lead to choosing the highest price $p = 1$ but instead leads to choosing an intermediate price. Note that for $a \in (0, 0.15]$ the profits are maximized at the highest price $p = 1$. This shows that the monotonicity result that Fabra and Llobet (2022) obtained for $a = 0$ is only partially robust to elastic demand: for any $a > \frac{3}{20}$ it breaks down.