Competing against a socially concerned firm when capacities are limited

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Abstract

The government may regulate a market by obtaining partial ownership in a firm, which induces the firm to maximize a weighted sum of its profits and the social surplus. We study price competition of a private firm against such a socially concerned firm with the novel and realistic assumption of capacity constraints. We highlight how the aggressive pricing of a publicly owned firm may induce the private firm to *in*crease its price. In contrast to other results in the literature that abstract from capacity constraints, we find that full privatization is the socially best outcome, that is the optimal level of public ownership is equal to zero when the semi-public firm has a smaller capacity level than the private firm. However, when the private firm is smaller than the semi-public firm, there is a positive optimal public ownership, which we characterize explicitly.

Keywords:

pricing, capacity constraints, public ownership, mixed duopoly, price competition.

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1 Introduction

In many markets, purely private firms compete with firms that are partially or wholly owned by the government. Such publicly owned companies often take consumer surplus into account, while privately owned firms set prices to maximize profits. A recent literature analyzed the question of optimal government ownership but under the assumption that firms can produce without capacity constraints. However, in many industries, including the natural gas and electricity industries, air and rail transportation, and pharmaceuticals the public-private competition takes place under capacity constraints. For example, in the case of LNG (Liquefied Natural Gas) markets capacity bottlenecks are present at least in the short to medium run due to the limited capacity of LNG terminals that receive the liquefied gas supplies. Many of the current competitors and planned new terminals in the European LNG sector are (or will be) partially owned by governments including the German Federal Government, France, Finland and the German State of Baden-Württemberg.[1] In the air travel industry, several leading airlines including Aeroflot, Air China, Air New Zealand, Finnair, ITA Airways, LOT Polish Airlines are wholly or partially owned by governments. In airplane manufacturing, Airbus whose 26% is owned by European governments,[2] is competing with privately owned Boeing. Lei (2016) indicates that this industry is capacity constrained and the two firms vigorously compete in prices with Airbus being somewhat more aggressive in pricing.[3]

Governments all over the world consider the role of the public sector in several industries to improve supply security, and reaction time or simply to reduce consumer prices. However, such policies are costly, and their effects must be carefully analyzed in each situation. To study the effects of public ownership, we set up a model where two firms with homogenous goods compete in prices under limited capacities. The government chooses its ownership share in one of the two firms (the semi-public firm) to maximize social surplus, while the other firm is fully private. The semi-public firm maximizes the convex combination of its profits and the social welfare with the weights depending on the public share in an increasing manner.

Our model is a Bertrand-Edgeworth game where firms set prices taking their capacities as given, and the firm with the higher price obtains a rationed, leftover demand.[4] Although price setting games with homogenous products fit several markets well, they are used less often due to technical complications arising from discontinuities in the payoff functions, which often prevent the existence of pure strategy equilibria. One of our contributions is deriving a novel form of the mixed strategy equilibrium that takes into account the incentives of the semi-public firm under capacity constraints, and the optimal reaction of the private firm.[5]

[1] Leading LNG firm GRTGaz is partially controlled by the French government through company Engie. New German LNG projects are run by partly publicly owned giants Uniper and RWE (Finnish and German shares), and utility company EnBW -owned mostly by the German public sector. On the other hand, other European players like Fluxys and Snam are fully private.

[2] At the time of its foundation in 1970 the governmental share in Airbus was above 80%, while its market share increased steadily.

[3] The fierce price competition through discounts is discussed in the Wall Street Journal article *The Secret Price of a Jet Airliner*. In line with our findings, that in case of capacity constraints the semi-public firm sets lower prices, when it comes to the lower range the Airbus 320 was slightly cheaper than the comparable Boeing 737-800, while for the longer range the Airbus 330 was significantly cheaper than its direct competitor Boeing 777.

[4] We assume efficient rationing for most of our analysis but also consider proportional rationing as an extension.

[5] Interestingly, we show that if the semi-public firm has a lower (or not much higher) capacity than the private

We find that the purely private price-setting duopoly maximizes social surplus when the semi-public firm has lower capacities than the private firm. This finding is in contrast with Matsumura (1998) and Barcéna-Ruiz and Sedano (2011) who assume no capacity constraints. Intuitively, an increase in the government's share in the semi-public firm induces the semi-public firm to lower its price to reflect the higher share of consumer surplus in its objectives. However, in response to such a price decrease, the private firm gives up on competing, and sets a price closer to the monopoly price on its residual demand curve. The reason this second (strategic) effect outweighs the direct effect is that social surplus depends more on the higher of the two prices than it depends on the lower of the prices because extra purchases are hard to make at the more congested cheaper firm.[6] Such a congestion effect does not arise when firms do not have capacity constraints, hence our stronger results.

When the semi-public firm is larger in capacities, the optimal level of public ownership is strictly positive. The optimal level is such that it still induces the private firm to compete vigorously without making the semi-public firm so strong as to induce the private firm to give up on competing and to set its monopoly price. At the optimal level of public ownership, the private firm's profits are reduced to its monopoly profits on the residual demand curve it faces. Any higher value of public ownership makes the private firm increase its price, which in turn reduces social surplus. In contrast, acquiring (or increasing) public ownership in a market leader can increase social welfare by reducing the incentives of such a firm to use its large market power to profit from high prices.

This insight is confirmed in a large number of markets where the government obtains public ownership in larger companies. This is the case in the airplane industry where Airbus has become the market leader in prices and quantities. In the infrastructure industry, it has also been increasingly common that the government sponsors larger projects that together reach a critical weight; see the discussion of the LNG market above. However, the government often acquires shares in smaller banks or pharmaceuticals that cannot be considered market leaders. In those cases, either the intervention is for another reason than to reduce prices, there is an intention to increase capacities of the semi-public firm or perhaps the policy is ineffective itself.[7]

The rest of this article is organized as followed. Upon a brief literature review, we describe the framework (Section 3), characterize the equilibrium (Section 4), and provide the welfare analysis (Section 5). In Section 6, we study an example with a different rationing rule, and in Section 7 we conclude. Most of the proofs are in the Appendix, while a worked out example with linear demand is provided in an Online Appendix.

firm, the private firm places an atom at the residual monopoly price and the support of its price features a gap below that monopoly price.

[6]However, the strategic effect itself may be smaller when the rationing rule is changed, an idea we further study in our Discussion Section.

[7]There is an extensive literature on privatization and nationalization. For a brief and informative overview, we refer to Goodman and Lovemen (1991) who list further factors like cost-effectiveness, governmental revenues through selling its shares, or decreasing the size of the state in favor of privatization.

2 Literature review

First, we review the literature of mixed duopolies in the absence of capacity constraints. In their seminal paper, Merrill and Schneider (1966) investigated the welfare effect of a public firm in a quantity-setting oligopoly. The case of a semi-public firm with an objective function obtained as a weighted sum of firm's profit and social surplus was analyzed by Matsumura (1998) in a homogeneous good quantity-setting duopoly. He determined the optimal governmental share and found an interior solution, that is the pure public firm case and the standard profit-maximizing case do not emerge as an optimal solution and a governmental share in a firm is beneficial. Similar investigations have been carried out for the heterogeneous goods price-setting duopoly game by Barcéna-Ruiz and Sedano (2011) in which again the optimal governmental share was positive.[8]

Our result is different from the case where firms are competing in quantities (as in Matsumura, 1998) or if products are differentiated (as in Barcéna-Ruiz and Sedano, 2011); in both those models competition is imperfect and public ownership helps in reducing prices. In our model, lower prices are enforced in the purely private duopoly game through the incentive of undercutting the opponent's price, an incentive that is weakened by semi-public ownership. Such an incentive for privately owned firms to compete by undercutting is absent with quantity-competition, and is reduced with differentiated products.

Closest to our paper in assuming capacity constraints on technology, Zhou et al. (2023) determine the optimal governmental share in a semi-mixed duopoly in which there is quality-differentiation, queuing of costumers and congestion in the consumption of goods. They propose their main applications in service industries like health care while our main area is the manufacturing and energy where rationing is more prevalent than queueing. Just like our work, Zhou et al. (2023) also takes capacities as given but, unlike our paper, they focus on the more tractable case of quantity competition. In our different framework, we are able to solve the more commonly observed case where firms set prices and not quantities. Their main finding is similar to ours in that they find that full privatization is the optimal solution if the customers are delay sensitive; however they can show this statement only numerically. Their result that a fully public firm is not welfare enhancing also involves a strategic effect where the private firm reacts in an adverse way to the more aggressive strategy (in terms of quantities) of the public or semi-public firm, although the exact mechanism of this strategic effect is quite different from ours.[9] The simpler mixed duopoly game with a purely public firm was investigated by Balogh and Tasnádi (2012) for which they found that an equilibrium in pure strategies always exists in contrast to the duopoly with a purely private firm, henceforth referred to as the standard case. However, since in the semi-public setting both firms' objective functions have a profit component, there is a capacity region for which a pure-strategy equilibrium does not exist. Hence, the analysis of the semi-public case becomes much more difficult.

Energy markets fit our stylized model due to the role of public ownership and capacity

[8] Casadesus-Masanell and Ghemawat (2006) and Zhou and Choudhary (2022) study models of open source software in a mixed duopoly.

[9] In a related paper, Hua et al. (2016) study the role of production subsidies to the public firm instead of the effects of (the level of) privatization.

constraints as it is also highlighted by the literature. For example, the capacity-constrained Bertrand-Edgeworth model is used in the modeling of energy markets (e.g. Vickers and Yarrow, 1990) and mixed duopoly models are also employed in energy markets (e.g. Escrihuela-Villar et al., 2020). The more recent work of Fabra and Llobet (2021) emphasize the role of capacity constraints under uncertain capacity levels. Due to recent instabilities caused by deteriorating supply conditions, the role of governments in energy markets has become an important question in European countries. Our paper informs this discussion by emphasizing the role of capacity constraints in determining the welfare effects of public-private competition.

Finally, our work is part of the Bertrand-Edgeworth literature. Because payoff functions are discontinuous at the point where firms set the same price,[10] the existence of equilibrium has been established by Maskin (1986) only in mixed strategies. Unfortunately, determining the equilibrium of a Bertrand-Edgeworth game requires complicated calculations even under restrictive assumptions. In duopoly models with linear demand functions with the same unit costs up to capacity constraints, Beckman (1965) determined the equilibrium in mixed-strategies under the proportional rationing rule. The case of efficient rationing was solved by Levitan and Shubik (1972) also for two firms.[11]

3 The framework

There are two firms with capacities $k_1, k_2 > 0$ who produce homogenous products and compete in prices. We denote the set of firms by $\{1, 2\}$, where 1 is the semi-public firm and 2 is the private firm. The industry faces a downward sloping demand function D that satisfies the following standard assumption:

Assumption 1. (i) D intersects the horizontal axis at quantity a and the vertical axis at price b; (ii) D is strictly decreasing and twice-continuously differentiable on $(0, b)$; (iii) D is right-continuous at 0 and left-continuous at b; (iv) $D(p) = 0$ for all $p \geq b$; and (v) $pD'(p)$ is strictly decreasing.

Note that the condition that pD' is strictly decreasing is equivalent to total welfare (surplus) being concave in p, while it is weaker than the concavity of D and stronger than the concavity of the revenue function pD. This assumption ensures that both objective functions π_1 and π_2 are concave in p and thus the problem is well-behaved.

Production costs are neglected except that firms cannot produce more than their capacities:

Assumption 2. The firms face zero unit cost up to their capacity constraints k_1 and k_2.[12]

Since for the interesting price region the low-price firm cannot satisfy the whole demand, its consumers have to be rationed so that the residual demand of the high-price firm is a function of the consumers served by the low-price firm. The most frequently employed rationing rule is the so-called efficient rationing rule, which is reasonable if there is a secondary market for the duopolists' products. In what follows $p_1, p_2 \in [0, b]$ stand for the prices set by the firms.

[10]The theory of supermodular games cannot be applied to obtain existence of equilibrium in pure strategies.

[11]Kreps and Scheinkman (1983) and Osborne and Pitchik (1986) relaxed the conditions on the demand function.

[12]The main assumption here is that firms have identical unit costs, assuming zero unit costs is just a matter of normalization since firms will produce to order.

Assumption 3. We assume efficient rationing on the market; that is, the demand faced by the firms $i \in \{1,2\}$ equals

$$\Delta_i\left(D,p_1,k_1,p_2,k_2\right)=\begin{cases} D\left(p_i\right) & \text{if} \quad p_i<p_j, \\ \frac{k_i}{k_1+k_2}D\left(p_i\right) & \text{if} \quad p=p_i=p_j, \\ \left(D\left(p_i\right)-k_j\right)^+ & \text{if} \quad p_i>p_j. \end{cases}$$

The assumption of efficient rationing is widely used for its tractability in the literature, and it means that the consumers with the highest valuations buy at the cheaper store first.[13]

We turn to specifying the firms' payoff functions. Recall that social surplus is equal to the sum of consumer surplus and profits in a single market, which is also the area under the demand curve in our case with zero costs. Let $\alpha \in (0,1)$ be the weight of the social surplus-maximizing component in the payoff function of the semi-public firm, which might be a function of the governmental share in the equity of firm 1. The extreme cases of $\alpha=0$ and $\alpha=1$ correspond to the already analyzed cases of the standard Bertrand-Edgeworth game and to the mixed version of the Bertrand-Edgeworth game investigated by Balogh and Tasnádi (2012). Let P denote the inverse demand function, that is $P\left(q\right)=D^{-1}\left(q\right)$ for $0<q\le a$, $P\left(0\right)=b$, and $P\left(q\right)=0$ for all $q>a$. Denote the residual demand curves of firm i by $D_i^r(p)=(D(p)-k_j)^+$, and denote the inverse of these residual demand curves by R_1 and R_2. The payoff function of the semi-public firm is given by

$$\begin{aligned} \pi_1(p_1,p_2) &= (1-\alpha)p_1\min\left\{k_1,\Delta_1\left(D,p_1,k_1,p_2,k_2\right)\right\}+ \\ &\quad \alpha\int_0^{\min\left\{(D(p_j)-k_i)^+,k_j\right\}}R_j(q)dq+\alpha\int_0^{\min\{a,k_i\}}P(q)dq, \qquad (1) \\ &= (1-\alpha)p_1\min\left\{k_1,\Delta_1\left(D,p_1,k_1,p_2,k_2\right)\right\}+ \\ &\quad \alpha\int_0^{D(p)}P(q)dq \qquad (2) \end{aligned}$$

where $0\le p_i\le p_j\le b$ and $p=p_j$ if $D(p)-k_i>0$, otherwise $p=p_i$. Observe that because of efficient rationing, social surplus is only a function of the largest price at which sales are realized. The private firm's payoff is equal to its profits:

$$\pi_2(p_1,p_2)=p_2\min\left\{k_2,\Delta_2\left(D,p_1,k_1,p_2,k_2\right)\right\}. \qquad (3)$$

Upon describing the setup, we define some useful price levels that characterize the incentives of the two firms. Let $p^c=P(k_1+k_2)$ the market clearing price, and by p^M the price set by a monopolist without capacity constraints, and by p_i^M the price set by a monopolist with capacity constraint k_i, where $i\in\{1,2\}$, i.e. $p^M = \arg\max_{p\in[0,b]} pD\left(p\right)$, and $p_i^M = \arg\max_{p\in[0,b]} p\min\{D\left(p\right),k_i\}$.

For $i\in\{1,2\}$ let

$$p_i^m=\arg\max_{p\in[0,b]} pD_i^r\left(p\right)$$

be the unique revenue maximizing price on the firms' residual demand curves $D_i^r(p) = (D(p)-k_j)^+$, where $j\in\{1,2\}$ and $j\neq i$, if $D_i^r(0)>0$. Let $p_i^m=0$ if $D_i^r(0)=0$. Clearly, p^c

[13] In case of equal prices we assume for simplicity that firms split demand in proportion to their capacities. However, we could have admitted a large class of tie-breaking rules, the only tie-breaking rules that have to be avoided are the ones that give full priority to one of the two firms.

and p_i^m are well defined whenever Assumptions 1 and 2 are satisfied. We have $p_i^M \geq p^M > p_i^m$. Furthermore, $k_1 < a$ implies $p_2^m > 0$. It can be easily verified that from $k_i > k_j$ it follows that $p_i^m > p_j^m$. We will assume that

$$k_i \leq D(p_i^m) \tag{4}$$

to cut down on the number of cases to be studied. This assumption holds when k_1 is smaller or not much larger than k_2.[14] This assumption ensures that any price in the support of equilibrium strategies, each firm cannot serve the entire demand when it is the low priced firm. This assumption is only for convenience, as the other case just requires studying the profit functions of the two firms piecewise[15] but the analysis (and the results) do not change in an important way.

Let us denote by p_i^d the smallest price p_i for which $p_i \min\{k_i, D_i(p_i)\} = p_i^m D_i^r(p_i^m)$, whenever this equation has a solution.[16] Provided that the private firm has 'sufficient' capacity (i.e. $p^c < p_2^m$), then the private firm is indifferent between serving residual demand at price level p_i^m or selling $\min\{k_i, D_i(p_i^d)\} = k_i$ at the lower price level p_i^d.[17] By Deneckere and Kovenock (1992, Lemma 1) we know that $p_i^d > p_j^d$ if $k_i > k_j$. We define the payoff maximizing price p_1^s for the semi-public firm when it faces residual demand:

$$p_1^s = \arg \max_{p_1 \in [0,b]} \left\{ (1-\alpha) p_1 D_1^r(p_1) + \alpha \int_0^{D(p_1)} P(q) dq \right\}.$$

It can be checked that p_1^s is determined uniquely and that $p_1^s < p_1^m$ under Assumptions 1-3.

4 Equilibrium existence and characterization

Concerning the pure-strategy equilibrium of the capacity constrained Bertrand-Edgeworth game with a socially concerned firm, henceforth called the semi-public Bertrand-Edgeworth game, the following holds:

Proposition 1. *Under Assumption 1-3, the semi-public Bertrand-Edgeworth game has a pure-strategy equilibrium if and only if* $\max\{p_1^s, p_2^m\} \leq p^c$. *If a pure-strategy equilibrium exists, then it is given by*

$$p_1^* = p_2^* = p^c = P(k_1 + k_2). \tag{5}$$

Proof. First, we show that whenever a pure-strategy equilibrium exists it can only be given by (5). Suppose that $p_1^* < p_2^*$. We start with the case of $D(p_1^*) > k_1$. If $D(p_2^*) > k_1$, then the semi-public firm can increase its profit by increasing its price such that social surplus will not change. If $D(p_2^*) \leq k_1$, then the private firm can gain profits by decreasing its price sufficiently. Turning to the case of $D(p_1^*) \leq k_1$, the private firm can make again profits by decreasing its price if $D(0) > k_1$ If $p_1^* = 0 < p_2^*$ and $D(0) = k_1$, then the semi-public firm can gain by increasing its

[14]For example when $D = 1 - p$, this condition boils down to $k_1 \leq (1 + k_2)/2$.

[15]In the two regions, the residual demand would satisfy $D_2^r = D - k_1$ or $D_2^r = 0$. This would only affect the form of firm one's price distribution strategy F but would not affect profits.

[16]The equation defining p_i^d has a solution if, for instance, $p_i^m \geq p^c$, which will be the case in our analysis when we will refer to p_i^d.

[17]This equality follows from assumption (4).

price since the monopoly price for the semi-public firm is positive (as it can be verified). Hence, an equilibrium in which $p_1^* < p_2^*$ does not exist.

Showing that in a pure-strategy-equilibrium, we cannot have $p_1^* > p_2^*$ is a bit simpler. If $p_1^* > p_2^* > 0$, then in case of $D(p_2^*) > k_2$ the private firm can sell its entire capacity at prices above p_2^*, while in case of $D(p_2^*) \leq k_2$ the semi-public firm can increase its payoff by setting a price below p_2^* since this will not change social surplus, while it can earn profits. If $p_1^* > p_2^* = 0$, then private firm can gain from increasing its price.

Thus, in a pure-strategy equilibrium both firms must set the same price $p_1^* = p_2^*$. However, there cannot be an equilibrium with $p_1^* = p_2^* > p^c$ because in this case at least one firm can benefit from unilaterally undercutting its opponent price. Clearly, $p_1^* = p_2^* < p^c$ cannot be an equilibrium neither.

Finally, by the concavity of the residual payoff functions and the definitions of p_1^s and p_2^m it follows that $p_1^* = p_2^* = p^c$ is a pure-strategy equilibrium if and only if $\max\{p_1^s, p_2^m\} \leq p^c$. □

The existence of a mixed-strategy equilibrium can be established by employing a recent existence theorem demonstrated by Prokopovych and Yannelis (2014, Theorem 3).

If a pure-strategy equilibrium exists, the standard, the mixed and the semi-public Bertrand-Edgeworth games all result in the same outcome in which the firms produce at their capacity constraints and the equilibrium price is the market clearing price.

Now, we turn to characterizing the mixed strategy equilibrium. The goal is twofold: besides getting a better understanding of the properties of the mixed-strategy equilibrium for the case of general demand functions, the results below also help characterizing the mixed-strategy equilibrium derived in the next section.

To make matters interesting, we assume that a pure-strategy equilibrium does not exist, i.e. $\max\{p_1^s, p_2^m\} > p^c$. We shall denote by (φ_1, φ_2) an arbitrary mixed-strategy equilibrium. Let $\overline{p}_i = \max \operatorname{supp}(\varphi_i)$ and $\underline{p}_i = \min \operatorname{supp}(\varphi_i)$, where $i \in \{1, 2\}$. Observe that $p_2^m > p^c$ implies $\underline{p}_2 \geq p_2^d > p^c$ because the private firms profits at price p_2^m are at least as large as at price p_2^d. Hence, $\underline{p}_1 \geq p_2^d$. Furthermore, if $p_1^s > p^c \geq p_2^m$, then $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$.

We present several results concerning the mixed-strategy equilibrium.[18] Lemma 1 shows that ties cannot occur with a positive probability, which derives from the fact that each firm would like to undercut the price of the other, a result that shows that competition is intense in prices.

Lemma 1. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$*, we obtain that* φ_1 *and* φ_2 *cannot both have an atom at the same price.*

The next two results characterize the upper bounds of the equilibrium price distributions. It shows that the firm with higher upper bound chooses a price that maximizes its payoffs conditional on losing the price war for sure. For firm two, this implies maximizing profits on the residual demand curve ($D_2^r = D(p_2) - k_1$), and for firm one it means maximizing its weighted objective of profits on the residual demand and consumer surplus.

[18]The proofs are in the Appendix.

Lemma 2. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$*, for any mixed-strategy equilibrium* (φ_1, φ_2) *we have* $\overline{p}_1 = p_1^s > \overline{p}_2$*,* $\overline{p}_1 < \overline{p}_2 = p_2^m$ *or* $\min\{p_1^s, p_2^m\} \leq \overline{p}_1 = \overline{p}_2 \leq \max\{p_1^s, p_2^m\}$*.*

Next, we show that the lower bounds of the equilibrium price distributions are identical, and feature no atoms.

Lemma 3. *Let Assumptions 1, 2, and 3 be satisfied and let* (φ_1, φ_2) *be a mixed-strategy equilibrium. If* $\max\{p_1^s, p_2^m\} > p^c$*, then* $\underline{p}_1 = \underline{p}_2$ *and* $\varphi_1(\underline{p}_1) = \varphi_2(\underline{p}_2) = 0$*.*

We have the following key result that partially characterizes the two possible of forms of the equilibrium:

Lemma 4. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$*, for any mixed-strategy equilibrium* (φ_1, φ_2) *it holds that one of the following two cases obtains in equilibrium:*
i) $\overline{p}_2 = p_2^m \geq \overline{p}_1$*;*
ii) $\overline{p}_1 = p_1^s \geq \overline{p}_2$*.*

Taking Lemma 4, we are ready to state the equilibrium characterization result:

Proposition 2. *When* $k_1 \leq k_2$ *the equilibrium is of the form i) for any* $\alpha \geq 0$*.*

When $k_1 > k_2$ *there exists* $\alpha^* > 0$ *such that the equilibrium is of the form i) when* $\alpha > \alpha^*$ *and the equilibrium is of the form ii) when* $\alpha \leq a^*$*. When* $\alpha \leq \alpha^*$ *it holds that* $\overline{p}_2 = p_1^s$*, and when* $\alpha > \alpha^*$ *it holds that* $\overline{p}_2 = p_2^m > \overline{p}_1 > p_1^s$*.*

Proof. We prove this result for the more complex case where $k_1 > k_2$, and argue at the end of the proof how the result for the case where $k_1 \leq k_2$ is derived. We start with some necessary conditions for an equilibrium of form ii). We then characterize a special case where one inequality condition holds as an equality, and calculate a value $\alpha = \alpha^*$ for that to be the case. When $\alpha = \alpha^*$ an equilibrium in form ii) holds by construction. We show that an equilibrium in form ii) holds also when $\alpha < \alpha^*$. Then we show that an equilibrium of form i) exists when $\alpha > \alpha^*$. Finally, we show that no other type of equilibria exist in either of the two regions.

An equilibrium of form ii) has by construction $\pi_1 = \pi_1^s$ and thus firm two does not place an atom at p_1^s. Let G denote the distribution function of p_2. Hence

$$\pi_1 = (1-\alpha)p[G(p)(D(p) - k_2) + (1 - G(p))k_1] +$$
$$+\alpha[G(p)SW(p) + \int_p^{p_1^s} g(x)SW(x)dx].$$

Similarly to the baseline case where $\alpha = 0$, it is easy to show that both firms randomize on the same interval $[\underline{p}, p_1^s]$ and thus $\pi_1'(p) = 0$ on that interval. Using that $SW'(p) = pD'(p)$, we obtain

$$0 = (1-\alpha)[G(D - k_2) + (1 - G)k_1 + pg(D - k_1 - k_2) + GpD'] + \alpha GpD'$$

or

$$g(p) = G'(p) = -\frac{k_1 + G(p)(D(p) - k_1 - k_2)}{p(D(p) - k_1 - k_2)} - \frac{G(p)D'(p)}{(1-\alpha)(D(p) - k_1 - k_2)}.$$

After rearranging, this becomes

$$G'(p) = -\frac{G(p)}{p} - \frac{k_1}{p(D(p)-k_1-k_2)} - \frac{G(p)D'(p)}{(1-\alpha)(D(p)-k_1-k_2)}. \tag{6}$$

Solving this differential equation with end condition $G(p_1^s) = 1$ yields a unique solution by the fundamental theorem of ordinary differential equations. (It is possible to solve this ODE explicitly but not necessary or useful for our purposes.) Then the lower bound of the strategy of firm two is at $\underline{p}(\alpha)$ s.t. $G(\underline{p}(\alpha), \alpha) = 0$ in the unique solution. Then the strategy of firm one, price distribution F, can be easily found by setting $F(p)$ such that for all $p \in [\underline{p}(\alpha), p_1^s]$,

$$p[F(p)(D(p)-k_1) + (1-F(p)k_2] = \underline{p}k_2 = \pi_2.$$

In order for this construction to be an equilibrium we need to show that neither firm one nor firm two can profitably deviate by setting any price larger than p_1^s. This holds by the definition of p_1^s for firm one. For firm two this condition becomes

$$\pi_2(\alpha) \leq \pi_2^m.$$

By the results of the literature, this holds as an inequality when $\alpha = 0$. Moreover, this condition fails when $\alpha = 1$ because in this case $p_1^s = 0$. Let

$$\alpha^* = \min(\alpha : \pi_2(\alpha) = \pi_2^m).$$

By construction, there exists an equilibrium in form ii) when $\alpha \leq \alpha^*$. The fact that $\pi_2(\alpha^*) = \pi_2^m$ implies that $F(p_1^s, \alpha^*) < 1$, and by the above argument $G(p_1^s, \alpha^*) = 1$.

Next, we show that for all $\alpha > \alpha^*$ there exists an equilibrium of type i). For this, take the differential equation (6) with the initial condition $\underline{p} = p_2^d = \pi_2^m/k_2$. When solving for the candidate equilibrium there can be two cases. First, it can hold that $\overline{p}_1 = \overline{p}_2 = p_2^m$ but together with the initial condition $\underline{p} = p_2^d$ this would lead to an overdetermined system as our argument below will imply. Therefore, $\overline{p}_1 = p_0 < p_2^m$. A smooth pasting condition imply that $G'(p_0) = g(p_0) = 0$ at that point otherwise firm one would have an incentive to deviate from choosing price p_0. Now, solving the initial value problem consisting of (6) with the initial condition $\underline{p} = p_2^d$, we can calculate $p_0(\alpha)$ that solves $G'(p_0(\alpha), \alpha) = 0$. By construction, due to the coinciding of region i) and region ii) solutions when $\alpha = \alpha^*$, it follows that $p_0(\alpha^*) = p_1^s$ and thus $G(p_0(\alpha^*), \alpha^*) = 1$.

We show now that for all $\alpha > \alpha^*$, $G(p_0(\alpha), \alpha) < 1$. For this, it is sufficient to prove that in the solution $G = 1 \Rightarrow G' < 0$ because $G = 0 \Rightarrow G' > 0$ and then by continuity a p exists such that $G < 1$ and $G' = 0$. To show that $G = 1 \Rightarrow G' < 0$, note that $G = 1$ implies that

$$G'(p) = -\frac{1}{p(D(p)-k_1-k_2)}\left(D(p) - k_1 - k_2 + k_1 + \frac{pD'(p)}{1-\alpha}\right).$$

Given that $p_0(\alpha^*) = p_1^s$ and thus $D(p_1^s) - k_1 - k_2 + k_1 + \frac{p_1^s D'(p_1^s)}{1-\alpha^*} = 0$. We show it in Lemma 5 that G is decreasing in α, and thus $G(p, \alpha) = 1$ implies that $p > p_1^s$ and thus

$$\begin{aligned} D(p) - k_1 - k_2 + k_1 + \frac{pD'(p)}{1-\alpha} &\leq D(p) - k_1 - k_2 + k_1 + \frac{pD'(p)}{1-\alpha^*} < \\ &< D(p_1^s) - k_1 - k_2 + k_1 + \frac{p_1^s D'(p_1^s)}{1-\alpha^*} = 0. \end{aligned}$$

Then $D(p) - k_1 - k_2 < 0$ implies that $G' < 0$ indeed holds.

It is clear that firm two does not have an incentive to use any price $p > p_0$ and $p \neq p_2^m$. However, we need to show that firm one does not have an incentive to choose a price $p > p_0$. For any such price price $g(p) = 0$ by construction and thus

$$\pi_1' = (1-\alpha)[G(p_0)(D(p) - k_2) + (1 - G(p_0))k_1] + G(p_0)pD'(p).$$

Given that pD' is decreasing in p by assumption, it follows that $\pi_1'' < 0$. Then using that (by smooth pasting) $\pi_1'(p_0) = 0$ we obtain that $\pi_1'(p) < 0$ for all $p > p_0$.

Given the above, there is an equilibrium when $\alpha > \alpha^*$ such that firm two randomizes on $[p_2^d, p_0]$ without any atoms and places an atom (with probability $1 - G(p_0(\alpha), \alpha)$) on price p_2^m. Firm one's strategy can be easily calculated from the incentive condition of firm two so that $\pi_2 = \pi_2^m$. Firm one places an atom at p_0 and uses equilibrium price support $[p_2^d, p_0]$. The calculation of F is similar as in region ii) and is omitted for brevity.

We have shown that there is an equilibrium in form i) when $\alpha > \alpha^*$, and there is an equilibrium in form ii) when $\alpha \leq \alpha^*$. We still need to rule out that there is an equilibrium in form i) when $\alpha \leq a^*$ or that there is an equilibrium of form ii) when $\alpha > \alpha^*$. First, we argue that a type i) equilibrium does not exist when $\alpha < \alpha^*$. Lemma 5 implies that in a type i) equilibrium the distribution of p_2 satisfies $\partial G/\partial\alpha < 0$. Therefore, $G(p, \alpha) > G(p, \alpha^*)$. Moreover,

$$G'(p, \alpha) \overset{sgn}{=} k_1 + G(p, \alpha)\left[\frac{D'(p)p}{1-\alpha} + D(p) - k_1 - k_2\right] = \beta(p, \alpha).$$

Under our assumptions, β is decreasing in both p and α when fixing G. Therefore, $p < p_1^s, \alpha < \alpha^* \implies \beta(p, \alpha) \geq k_1 + [\frac{D'(p)p}{1-\alpha} + D(p) - k_1 - k_2] > \beta(p_1^s, \alpha^*) = 0$ where the equality follows from $G'(p_1^s, \alpha^*) = 0$, which follows from the definition of α^*.[19] The inequalities follow from the monotonicity of β and from the fact that on the relevant interval $G(p, \alpha) \leq 1$ and the term in the squared bracket is negative. Finally, notice that $G(p, \alpha) > G(p, \alpha^*)$ implies that the relevant prices include only prices such that $p < p_1^s$ because $G(p_1^s, \alpha^*) = 1$. Putting everything together implies that $G'(p, \alpha) > 0$ for all prices that firm two uses, which contradicts with smooth pasting.

For the solution in region ii) we show that when $\alpha > \alpha^*$ the induced profit of firm two satisfies $\pi_2(\alpha) < \pi_2^m$, which rules out such an equilibrium. Suppose that an equilibrium of type ii) exists and thus for some $\alpha > \alpha^*$ we have $\pi_2(\alpha) \geq \pi_2^m$. Then the lower bound of the prices satisfies $\underline{p}(\alpha) \geq p_2^d = \underline{p}(\alpha^*)$, which then implies that $0 = G(\underline{p}(\alpha), \alpha) \leq G(\underline{p}(\alpha), \alpha^*)$. Then comparing the first order conditions we obtain from Lemma 5 that $G(p, \alpha) < G(p, \alpha^*)$ for all $p > \underline{p}(\alpha)$. This then implies $G(p_1^s(\alpha), \alpha) \leq G(p_1^s(\alpha^*), \alpha) < G(p_1^s(\alpha^*), \alpha^*) = 1$ and then we have two cases.

a) If $\overline{p}_2 > p_1^s$ then ii) is violated.

b) If $\overline{p}_2 = p_1^s$, then the distribution of p_2, G places an atom at p_1^s. In this case, the distribution of p_1, F does not place an atom at p_1^s. Then since $p_2^m > p_1^s$ for all $\alpha > \alpha^*$,[20] we obtain that firm two is better off choosing p_2^m than choosing p_1^s, which yields a contradiction with an equilibrium of type ii).

[19] At $\alpha = \alpha^*$ the solution (both of the form i) and of the form ii)) involves $G(p_1^s) = 1$ and $G'(p_1^s) = 0$.

[20] Note, that $p_1^s \leq p_2^m$ holds at $\alpha = \alpha^*$ because $p_1^s > p_2^m$ would imply that $\pi_2 > \pi_2^m$ in a type ii) equilirbium, which contradicts with the fact that $\pi_2 = \pi_2^m$ when $\alpha = \alpha^*$ by construction. Since p_1^s is decreasing α, $p_1^s < p_2^m$ holds for all $\alpha > \alpha^*$.

When $k_1 \leq k_2$ a type i) equilibrium exists even if $\alpha = 0$. Then our uniqueness argument implies that only a type i) equilibrium exists for all $\alpha > 0$. □

As Deneckere and Kovenock (1992) show it when $\alpha = 0$ the larger firm prices less aggressively than the smaller. This result dictates whether i) or ii) applies when $\alpha = 0$. Our Proposition 2 extends their result when the smaller firm is also socially concerned ($k_1 \leq k_2$ and $\alpha > 0$), and thus has extra incentive to price aggressively. When $k_1 > k_2$, the two effects contradict each other, and the value of α determines which firm has lower prices.

In case of $k_1 \leq k_2$ let $\alpha^* = 0$ to capture the results of these two effects on the equilibrium price distribution produced by the private firm:

Lemma 5. *When $\alpha > \alpha^*$, $G(p, \alpha)$ is strictly decreasing in $\alpha \in [0, 1]$ for any $p \in (p_2^d, p_2^m]$. When $\alpha \leq \alpha^*$, $G(p, \alpha) < G(p, \alpha^*)$ for all p on the support of price distributions. Thus for any p on the support of a price distribution $G(p, \alpha)$ is maximized at $\alpha = \alpha^*$.*

Suppose that $k_1 > k_2$ and thus $\alpha^* > 0$. First, suppose that firm one is relatively weak in that α is low and thus firm one tends to be less aggressive in its pricing strategy. In this case, an increase in α makes firm one more aggressive, and induces firm two to compete harder, hence $\alpha \leq \alpha^* \implies G(p, \alpha) < G(p, \alpha^*)$. Second, when firm one has already a high welfare concern, a further increase in α discourages firm two from competing vigorously. Instead of competing, firm two starts setting his (residual) monopoly price p_2^m with a higher and higher probability as α goes further above α^*.

5 Welfare analysis

Lemma 5 has repercussions for the level of total welfare. For example, when firm one becomes fully public ($\alpha \to 1$),[21] firm two loses its chance to become the firm with the lower price. Therefore, since it is losing the price war anyway, firm two sets the monopoly price on the residual demand curve p_2^m in the limit. Furthermore, given that under the efficient rationing rule only the higher of the two prices matter for the social surplus, the fact that $p_2 \to p_2^m$ implies that social surplus is *minimized* when α approaches one and $k_1 \leq k_2$. We state the following result that shows that welfare is decreasing in α when the semi-public firm is smaller than the private firm:

Proposition 3. *When $k_1 \leq k_2$ total welfare is decreasing in α.*

The intuition for why public ownership decreases welfare when $k_1 \leq k_2$ is clear when α is close to one. The fact that it holds also for lower levels of α is not straightforward. On one hand, it is true that firm two is less aggressive when α is increasing as Lemma 5 shows for the case where $k_1 \leq k_2$ and thus $\alpha^* = 0$. On the other hand, one can expect that firm one prices more aggressively as α increases. We have shown this result in a worked out example for the case of linear demand and $k_1 = k_2$ in the Online Appendix. However, since firm two tends to charge

[21] Balogh and Tasnádi (2012) has analyzed this game under the assumption that $\alpha = 1$, and (depending on parameter values) obtained two or three pure-strategy equilibria. Our result in effect selects one of those equilibria as $\alpha \to 1$. In particular, their NE_2-type equilibrium is approached.

a higher price and welfare depends on the higher of the two prices, therefore total welfare is unambiguously reduced when the public component α increases.

We are ready to state our result on social welfare when the semi-public firm is larger than the private firm:

Proposition 4. *When $k_1 > k_2$ total welfare is maximized when $\alpha = \alpha^* \in (0, 1)$.*

Propositions 3 and 4 imply that as a way to increase competition, it is better to acquire public ownership in the market leader. Moreover, public ownership is only useful socially as long as it reduces the profits of the private company but increasing the public share any further ($\alpha > \alpha^*$) only serves to discourage the private firm from competing, and increases prices overall.[22] Instead of acquiring an even higher level of public ownership, investing in more capacities for the semi-public firm is more likely to serve the interest of the public better. The combination of an optimal capacity choice together with the choice of optimal public ownership is left for future research.

6 Discussion: the role of the rationing rule

We have identified two effects: the direct effect implies that the price of the semi-public firm decreases in α, while the strategic effect means that the price of the private firm increases in α due to its lower incentive to compete with the semi-public firm. These two effects generalize to other rationing rules, including the proportional rationing rule, the other major rationing rule used in the literature. The unique feature of the efficient rationing rule is that only the higher of the two price matters, which means that the social surplus is completely governed by the strategic effect. For a rationing rule that is similar to the efficient rule mathematically, we can still expect that the social surplus is decreasing in α. However, when the rationing rule is changed to a rule where social surplus depends on the lower price to a larger extent, the result may change.

To illustrate this possibility, take the case of linear demand and assume that rationing is proportional. Assuming $p_2 > p_1$, proportional rationing means that $D_2^r = (1 - \frac{k}{1-p_1})(1 - p_2)$. In this case, the equilibrium is in pure strategies when α is close to one for a certain region of capacities. We identify such a region, and show that social surplus is increasing in α in that region because $p_2 = p^M = 1/2$ and p_1 is decreasing in α.

In this case, the private firm maximizes $pD(p)(1 - \frac{k}{1-p_1})$, which is maximized at $p^M = 1/2$. The rest of the analysis first derives the best response of the semi-public firm (firm 1), and then we check back whether the private firm (firm 2) has an incentive to deviate.

When $\alpha = 1$ firm 1 maximizes social surplus. If firm 1 chooses $p_1 < p_2 = p^M = 1/2$, then all consumers with valuations higher than $1/2$ trade with probability one, and all consumers with valuations between p_1 and $1/2$ buy with probability $k/(1 - p_1)$. Therefore, social surplus, the sum of consumer and producer surplus is

$$SS = \frac{1}{2}\frac{3}{4} + \frac{k}{1-p_1}(\frac{1}{2} - p_1)\frac{1/2 + p_1}{2}$$

[22] For example, a fully nationalized company ($\alpha = 1$) always outcompetes its rival in prices, and thus the private firm may just as well set its (residual) monopoly price as noted above.

by using the gross consumer surplus formulation. The problem of firm 1 becomes

$$\max_{p_1} \frac{1/4 - (p_1)^2}{1 - p_1}$$

with solution $p_1 = 1 - \sqrt{3}/2 = 0.13$.

Now, let us check if firm 2 has an incentive to undercut firm 1's price. The equilibrium profit is $\frac{1}{4}(1 - \frac{k}{\sqrt{3}/2})$. The profit from undercutting and charging $p_1 - \varepsilon$ is equal to $kp_1 = k(1 - \sqrt{3}/2)$, and thus the incentive condition becomes $\frac{1}{4}(1 - \frac{k}{\sqrt{3}/2}) \geq k(1 - \sqrt{3}/2)$ or $k \leq \frac{1}{4 + \frac{2\sqrt{3}}{3} - 2\sqrt{3}} = 0.59151$. Focusing on the range where $k \leq 0.59151$ the equilibrium is in pure strategies when $\alpha = 1$. Suppose that k is (much) lower than this cutoff and thus there is a pure strategy equilibrium for a range where $\alpha < 1$. Analyzing the equilibrium in this case, it still holds that $p_2 = 1/2$. By construction, firm 1 maximizes social surplus when $\alpha = 1$. Given that firm 2 sets $p_2 = 1/2$ for any α it must be that in this range TW is maximized when $\alpha = 1$ given that there is no strategic effect that affects firm 2's choice.

In summary, in the range where α is high social surplus is maximized when $\alpha = 1$. The reason is the lack of strategic effect in this case: firm two sets the monopoly price in equilibrium regardless of α, which then implies that the direct effect through p_1 makes social surplus increase in α. When α is close to zero, the equilibrium is in mixed strategies, see Beckman (1965). In this case, a change in α changes the equilibrium price distribution of firm two, and further calculations are necessary to settle the question even for the case of linear demand.[23]

7 Conclusions

We have analyzed the problem of a semi-public firm competing with a private firm under capacity constraints. We have shown that under the commonly used efficient rationing rule, social surplus is decreasing in the public concern of the semi-public firm when the semi-public firm is smaller (or not much larger) than the private firm. We also provided the optimal level of public ownership when the semi-public firm has a larger capacity. We also highlighted that for general rationing rules there are two opposing effects. First, a higher level of α directly increases social surplus by the action of the semi-public firm. On the other hand, α also indirectly affects the price set by firm two as it was highlighted under efficient rationing. This second effect is ambiguous, and can be surplus reducing especially if the rationing rule is close to the efficient rationing rule. It remains for future research to study general conditions for comparative statics in terms of demand conditions and rationing rules.

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[23] Given that $p_2 = p^M$ when α is high, it must be the case that p_2 is increasing in α at least for some values of α. Therefore, the strategic effect of α on p_2 may counterbalance the direct effect and Proposition 3 may or may not hold.

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8 Appendix

Proof of Lemma 1: Suppose that there exists a price $p \in [0, b]$ for which $\varphi_1(p) > 0$ and $\varphi_2(p) > 0$. However, this would imply because of $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$ that both firms $i \in \{1, 2\}$ would be better off by unilaterally shifting probability mass from price p to $p - \varepsilon$; a contradiction. $\square$

Proof of Lemma 2: Let $\overline{p}_1 > \overline{p}_2$. If $\overline{p}_1 > p_1^s$, then the semi-public firm could benefit from setting a price below $\overline{p}_1$ because of the strict concavity of its residual payoff function. If $\overline{p}_1 < p_1^s$, then the semi-public firm would make more profits by setting price p_1^s than setting any other price in $(\overline{p}_2, p_1^s)$; a contradiction. Hence, in case of $\overline{p}_1 > \overline{p}_2$ we must have $\overline{p}_1 = p_1^s$.

In an analogous way it can be shown that if $\overline{p}_1 < \overline{p}_2$, we must have $\overline{p}_2 = p_2^m$.

Suppose that $\min\{p_1^s, p_2^m\} > \overline{p}_1 = \overline{p}_2$. Then since in equilibrium at least one of the mixed strategies cannot have an atom at $\overline{p}_1 = \overline{p}_2$, say $\varphi_i(\{\overline{p}_i\}) = 0$, firm $j \neq i$ could increase its payoff by setting either price p_1^s or p_2^m; a contradiction.

Suppose that $\overline{p}_1 = \overline{p}_2 > \max\{p_1^s, p_2^m\}$. Then since in equilibrium at least one of the mixed strategies cannot have an atom at $\overline{p}_1 = \overline{p}_2$, say $\varphi_i(\{\overline{p}_i\}) = 0$, and thus firm $j \neq i$ would serve residual demand with probability one at price $\overline{p}_j$ from which it follows that its payoff would be higher at price p_1^s or p_2^m (in the former case this would result for the semi-public firm both an increase in profit and social surplus); a contradiction. $\square$

Proof of Lemma 3: First, we establish that $\overline{p}_i \leq p_i^M$. Clearly, the semi-public firm's prices above p_1^M would be strictly dominated by price p_1^M (i.e. $\pi_1(p_1^M, \varphi_2) > \pi_1(p_1, \varphi_2)$ for any $p_1 > p_1^M$ and any mixed strategy φ_2 played by the private firm). The case of $\overline{p}_2 \leq p_2^M$ is even more obvious. Hence, the firms' do not set 'extremely' high prices.

Second, we demonstrate that $\underline{p}_1 \leq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 > \underline{p}_2$. Then by $\underline{p}_2 < \overline{p}_2 \leq p_2^M$ the private firm would benefit from switching from φ_2 to any price $p_2 \in (\underline{p}_2, \underline{p}_1)$; a contradiction.

Third, we demonstrate that $\underline{p}_1 \geq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 < \underline{p}_2$. Then by $\underline{p}_1 < \overline{p}_1 \leq p_1^M$ the public firm would benefit from switching from φ_1 to any price $p_1 \in (\underline{p}_1, \underline{p}_2)$ since the profit component of its payoff function would increase and the social surplus component of its payoff function would not change; a contradiction.

Forth, suppose that $\varphi_1(\underline{p}_1) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_1 - \varepsilon$ would strictly dominate price $\underline{p}_1 + \varepsilon$ for the private firm; a contradiction.

Finally, suppose that $\varphi_2(\underline{p}_2) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_2 - \varepsilon$ would strictly dominate price $\underline{p}_2 + \varepsilon$ for the semi-public firm since its profit component would be radically larger at the former price than at the latter one by its discontinuity at $\underline{p}_2$, while the social surplus component would be just slightly lower by the continuity of the social surplus component; a contradiction. $\square$

Proof of Lemma 4: Step 1. First, we show that $\overline{p} = \max\{\overline{p}_1, \overline{p}_2\} = p_1^s$ or p_2^m using Lemma 2. In particular, let i be the firm that does not place an atom at $\overline{p}$. Suppose that $i = 1$ first, and we show that $\overline{p}_2 = \overline{p} \geq p_2^m$. Suppose that $\overline{p}_2 = \overline{p} < p_2^m$, and notice that in this case it would be profitable for firm two to use p_2^m instead of $\overline{p}_2$ because both prices lose the price competition for sure, and thus p_2^m is more profitable by definition. When $i = 2$ a similar argument implies $\overline{p}_1 \geq p_1^s$.

Step 2. Next, we show that $\overline{p}_2 = \overline{p} > p_2^m$ when $i = 1$ and $\overline{p}_1 = \overline{p} > p_1^s$ when $i = 2$. Take $i = 1$, the other case proceeds similarly. When $\overline{p}_2 > p_2^m, \overline{p}_1$, it is clear that a deviation from $\overline{p}_2$ to $\max\{p_2^m, \overline{p}_1\}$ is profitable for firm two by the same reason as in the case of Step 1. Now, suppose that $\overline{p}_2 = \overline{p}_1 > p_2^m$. In this case, reducing the price from $\overline{p}_2$ to p_2^m increases profits for firm two both when $p_2 > p_1$ and when $p_2 > p_1$.

Combining the two steps yields the desired result. □

Proof of Lemma 5: First, we prove the result for the case where α is larger than α^*. Letting $\alpha_2 > \alpha_1 \geq \alpha^*$, our first step shows the following.

Step 1: It holds on a neighborhood $p \in (p_2^d, p_2^d + \varepsilon)$ that $G(p, \alpha_2) < G(p, \alpha_1)$.

By construction $G(p, \alpha_2) = G(p, \alpha_1) = 0$ when $p = p_2^d$. Therefore, (6) implies $G'(p_2^d, \alpha_2) = G'(p_2^d, \alpha_1) = -k/\left(p_2^d(D(p_1) - 2k)\right) > 0$. Suppose that no such ε exists. Then either $G(p, \alpha_2) = G(p, \alpha_1)$ on an entire neighborhood of p_2^d but that contradicts (6). Therefore, then $G(p, \alpha_2) > G(p, \alpha_1)$ on a neighborhood of p_2^d.

Next, we will show that on this neighborhood where $G(p, \alpha_2) > G(p, \alpha_1)$, it follows that $G'(p, \alpha_2) < G'(p, \alpha_1)$, which yields a contradiction because $G(p, \alpha_2) = G(p, \alpha_1) = 0$ when $p = p_2^d$. After rewriting (6),

$$G'(p_1) = -\frac{D'(p_1)G(p_1)}{(1-\alpha)\,(D(p_1) - 2k)} - \frac{k}{p_1(D(p_1) - 2k)} - \frac{G(p_1)}{p_1}.$$

Therefore, G' is decreasing in both α and G using $D', D - 2k < 0$. Therefore, $\alpha_2 > \alpha_1$ and $G(p, \alpha_2) > G(p, \alpha_1)$ imply that $G'(p, \alpha_2) < G'(p, \alpha_1)$, providing the required contradiction for our proof by contradiction for the result in Step 1.

Step 2. There is no $p > p_2^d$ where $G(p, \alpha_2) = G(p, \alpha_1)$.

Suppose there is such a p and take the smallest such value p^* (a minimum exists because G is continuous in p). Since $G(p, \alpha_2) < G(p, \alpha_1)$ for all $p \in (p_2^d, p^*)$, it must hold that $G'(p^*, \alpha_2) \geq G'(p^*, \alpha_1)$. But by the same argument as in Step 1, $G(p^*, \alpha_2) = G(p^*, \alpha_1)$ and $\alpha_2 > \alpha_1$ imply $G'(p^*, \alpha_2) < G'(p^*, \alpha_1)$, which leads to a contradiction establishing that $G(p, \alpha)$ is decreasing in α when $\alpha \geq \alpha^*$.

For the case, where $\alpha < \alpha^*$, notice that the lower end of the supports satisfy $\underline{p}(\alpha^*) = p_2^d > \underline{p}(\alpha)$ because $\pi_2(\alpha) < \pi_2^m$. This then completes Step 1 of the proof, establishing $G(p, \alpha) > G(p, \alpha^*)$ if p is close to $\underline{p}(\alpha)$.

Now, consider a hypothetical crossing point such that $G(\widehat{p}, \alpha) = G(\widehat{p}, \alpha^*)$. Then the same argument as above implies that $G'(\widehat{p}, \alpha) > G'(\widehat{p}, \alpha^*)$ and thus $G(p, \alpha) > G(p, \alpha^*)$ for all $p > \widehat{p}$. But $p_1^s(\alpha) > p_1^s(\alpha^*)$ and thus in a region i) solution $G(p_1^s(\alpha^*), \alpha^*) = 1$ but $G(p_1^s(\alpha^*), \alpha) <$

1 by $p_1^s(\alpha) > p_1^s(\alpha^*)$. Therefore, $G(p,\alpha) > G(p,\alpha^*)$ fails at $p = p_1^s(\alpha^*)$, which provides a contradiction, so such a crossing point cannot exist. □

Proof of Proposition 3: We shall denote by H the cumulative distribution function of the higher price. Hence, $H(p,\alpha) = \overline{F}(p,\alpha)\overline{G}(p,\alpha)$ for all $p \in [p_2^d, p_2^m]$ and for any given $\alpha \in [0,1]$, where we emphasize that each $\alpha \in [0,1]$ specifies a different game with the respective equilibrium strategies $\overline{F}(p,\alpha)$ and $\overline{G}(p,\alpha)$ of the two firms.

Then for any $p \in \left[p_2^d, p_0(\alpha)\right)$, where $g(p_0(\alpha),\alpha) = 0$, so p_0 is defined the same way as in in the proof of Proposition 2, we have

$$\frac{\partial H}{\partial p}(p,\alpha) = h(p,\alpha) = f(p)G(p,\alpha) + F(p)g(p,\alpha), \tag{7}$$

$$\frac{\partial H}{\partial \alpha}(p,\alpha) = F(p)\frac{\partial G}{\partial \alpha}(p,\alpha), \tag{8}$$

$$\frac{\partial^2 H}{\partial p \partial \alpha}(p,\alpha) = \frac{\partial h}{\partial \alpha}(p,\alpha) = f(p)\frac{\partial G}{\partial \alpha}(p,\alpha) + F(p)\frac{\partial g}{\partial \alpha}(p,\alpha), \tag{9}$$

where $g(p,\alpha) = (\partial G/\partial p)(p,\alpha)$.

Since for the relevant price region $[p_2^d, p_2^m]$ social surplus is determined by the higher price set by the two firms to prove our theorem it is sufficient to show that $H(\cdot,\alpha)$ first order stochastically dominates $H(\cdot,\alpha')$ for any $1 \geq \alpha > \alpha' \geq 0$. We establish this dominance relationship by showing that $(\partial H/\partial p)(p,\alpha) \leq 0$ for any $p \in [p_2^d, p_2^m]$. Clearly, this inequality holds for any $p \in [p_2^d, p_0(\alpha))$ by Lemma 5 and Equation (8). Therefore, we still have to consider the case of $p \in [p_0(\alpha), p_2^m)$ for which $H(p,\alpha)$ is given by

$$H(p,\alpha) = \int_{p_2^d}^{p_0(\alpha)} f(r)G(r,\alpha) + F(r)g(r,\alpha)dr + (1 - F(p_0(\alpha)))\, G(p_0(\alpha),\alpha).$$

By differentiating with respect to α,[24] thereafter by rearrangements and taking $g(p_0(\alpha),\alpha) =$

[24]Note that $F(p_2^d) = 0$ and $G(p_2^d,\alpha) = 0$.

$(\partial G/\partial p)\,(p_0(\alpha),\alpha)=0$ into account, and finally employing Young's theorem, we get

$$\begin{aligned}
\frac{\partial H}{\partial \alpha}(p,\alpha) &= \left[f(p_0(\alpha))G(p_0(\alpha),\alpha)+F(p_0(\alpha))g(p_0(\alpha),\alpha)\right]p_0'(\alpha)\\
&\quad+\int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial g}{\partial \alpha}(r,\alpha)dr\\
&\quad-f(p_0(\alpha))p_0'(\alpha)G(p_0(\alpha),\alpha)\\
&\quad+\left(1-F\left(p_0(\alpha)\right)\right)\left[\frac{\partial G}{\partial p}\left(p_0(\alpha),\alpha\right)p_0'(\alpha)+\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)\right]\\
&= \int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial g}{\partial \alpha}(r,\alpha)dr\\
&\quad+\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)\\
&= \int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial^2 G}{\partial \alpha\partial p}(r,\alpha)dr\\
&\quad+\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)\\
&= \left[F(r)\frac{\partial G}{\partial \alpha}(r,\alpha)\right]_{p_2^d}^{p_0(\alpha)}+\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)\\
&= \frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)<0.
\end{aligned}$$

□

Proof of Proposition 4:

The argument in the proof of Proposition 3 established that total welfare is strictly decreasing in α when $\alpha \geq \alpha^*$. In the case where $\alpha < \alpha^*$, Lemma 5 implies that $G(p,\alpha) < G(p,\alpha^*)$. Then a similar argument as in the proof of Proposition 3 can be constructed. In particular, take any function $\widetilde{G}$ such that $\widetilde{G}(p,\alpha)=G(p,\alpha)$ and $\widetilde{G}(p,\alpha^*)=G(p,\alpha^*)$ and $\widetilde{G}$ is increasing in α. Then the same argument as in the proof of Proposition 3 implies that social welfare is increasing in α for this modified problem, and in particular $TW(\alpha^*) > TW(\alpha)$, which concludes the proof. □

9 Online Appendix: An example with linear demand

In this Appendix, we assume that $D=1-p$ and $k_1=k_2=k$ to obtain a closed form solution. In this case, we have an equilibrium of type i). By differentiating π_1, we obtain

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1}(p_1,G) &= (1-\alpha)k(1-G(p_1))-(1-\alpha)p_1kg(p_1)\\
&\quad+(1-\alpha)\left[(1-p_1-k)G(p_1)-p_1G(p_1)+p_1(1-p_1-k)g(p_1)\right]\\
&\quad-\alpha p_1G(p_1)+\frac{1}{2}\alpha(1-p_1^2)g(p_1)\\
&\quad-\frac{1}{2}\alpha(1-p_1^2)g(p_1)\\
&= \left[(1-\alpha)(1-2p_1-2k)-\alpha p_1\right]G(p_1)\\
&\quad+(1-\alpha)p_1(1-p_1-2k)g(p_1)+(1-\alpha)k=0,
\end{aligned}$$

where g is the derivative of G, and the expression is just defined where G is differentiable. Solving the first-order linear differential equation, we get[25]

$$G(p_1) = C\frac{1}{p_1}\left(\frac{1}{2k+p_1-1}\right)^{\frac{1}{1-\alpha}} + \frac{k(1-\alpha)}{p_1} \tag{10}$$

and employing $G(p_2^d) = 0$ we arrive at

$$C = -k(1-\alpha)\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}.$$

Upon substituting C in (10), we obtain

$$G(p) = \frac{k}{p}(1-\alpha)\left(1 - \left(\frac{\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^2}{2k+p-1}\right)^{\frac{1}{1-\alpha}}\right). \tag{11}$$

We have the following result:

Claim. It holds that $\partial F/\partial\alpha > 0$ for all p on the support of $F(p,\alpha)$.

Proof. Let $p_0 = \overline{p}_1$ solve $G'(p_0) = 0$.

Step 1: We show that $\partial p_0/\partial\alpha < 0$.

Let $z = \frac{(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}})^2}{2k+p_0-1}$, and take the FOC $G'(p_0) = 0$ or using (11),

$$p_0\left(-\frac{1}{1-\alpha}z'z^{\frac{\alpha}{1-\alpha}}\right) = 1 - z^{\frac{1}{1-\alpha}}.$$

Also,

$$\begin{aligned}\frac{\partial G}{\partial\alpha} &= -\frac{k}{p_0}(1 - z^{\frac{1}{1-\alpha}}) + \frac{k}{p}(1-\alpha)\left(-\frac{z^{\frac{1}{1-\alpha}}\ln z}{(1-\alpha)^2}\right) = \\ &= -\frac{G}{1-\alpha} - \frac{k}{p}(1-\alpha)\frac{\ln z}{(1-\alpha)^2}\left(1 - \frac{G}{\frac{k}{p}(1-\alpha)}\right) = \\ &= -\frac{G}{1-\alpha} - \frac{\ln z}{(1-\alpha)^2}\left(\frac{k}{p}(1-\alpha) - G\right).\end{aligned}$$

Then using that $G'(p_0) = 0$, we obtain that

$$\begin{aligned}\frac{\partial^2 G}{\partial p\partial\alpha} &= -\frac{z'}{z(1-\alpha)^2}\left(\frac{k}{p_0}(1-\alpha) - G\right) + \frac{\ln z}{(1-\alpha)^2}\frac{k}{p_0^2}(1-\alpha) = \\ &= -\frac{z'}{z(1-\alpha)^2}\frac{k}{p_0}(1-\alpha)z^{\frac{1}{1-\alpha}} + \frac{\ln z}{(1-\alpha)^2}\frac{k}{p_0^2}(1-\alpha) \overset{sgn}{=} \\ &\overset{sgn}{=} \frac{\ln z}{p_0} - \frac{z'}{z}z^{\frac{1}{1-\alpha}} = \frac{\ln z}{p_0} - z'z^{\frac{\alpha}{1-\alpha}}.\end{aligned}$$

Upon using $p_0(-\frac{1}{1-\alpha}z'z^{\frac{\alpha}{1-\alpha}}) = 1 - z^{\frac{1}{1-\alpha}}$, we obtain

$$\frac{\partial^2 G}{\partial p\partial\alpha} \overset{sgn}{=} \frac{\ln z}{1 - z^{\frac{1}{1-\alpha}}}\left(-\frac{1}{1-\alpha}z'z^{\frac{\alpha}{1-\alpha}}\right) - z'z^{\frac{\alpha}{1-\alpha}} \overset{sgn}{=} \ln z + (1-\alpha)(1 - z^{\frac{1}{1-\alpha}}).$$

[25] Under our assumptions we have that $2k + p - 1 = p - p^c > 0$ since $p_1^d = p_2^d > p^c$.

upon using that $z' < 0$.

Therefore, we just need to establish that for all $z, \alpha \in (0, 1)$, it holds that

$$\omega = \ln z + (1 - \alpha)(1 - z^{\frac{1}{1-\alpha}}) < 0.$$

This holds as an equality at $z = 1$. Taking a derivative with respect to z, $\omega' = \frac{1}{z} - z^{\frac{1}{1-\alpha}-1} > 0$, and thus $\omega(1) = 0$ implies that $\omega(z) < 0$ for all $z < 1$. Then using the implicit function theorem and the second order condition for G at p_0 imply $\partial p_0/\partial \alpha < 0$.

Step 2. The rest of the proof just uses that for all $p \in [p_2^d, p_0)$, it holds that

$$p[F^*(p)(1 - p - k) + (1 - F^*(p))k] = \pi_2 = \pi_2^m.$$

It is clear that F does not depend on α only through p_0. In particular, $F(p) = F^*(p)$ if $p \in [p_2^d, p_0)$, and $F(p) = 1$ if $p \geq p_0$. Given that $\partial p_0/\partial \alpha < 0$, the result follows. $\square$

Discussion: When α increases, the semi-public places a larger atom on a smaller price p_0, while the rest of its price distribution is unaffected. This type of truncation delivers the monotonicity result as long as $\partial p_0/\partial \alpha < 0$.